Topological Closure on the Complexified Null Cone

A Heegaard Transgression Theorem and an Open Residue Problem in Twistor Theory

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Abstract

We study a local geometric-topological structure associated with the complexified null cone. Over $\mathbb{R}$, the Lorentzian null cone contains no real projective ruling lines; over $\mathbb{C}$, the corresponding smooth quadric is doubly ruled and admits the standard spinor factorisation $k_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$. Motivated by the positive/negative-frequency split in twistor theory, we examine a two-sector interpretation of the two ruling families and the associated local sky-bundle geometry.

For a closed loop $\gamma \subset S^2$ in the space of null directions, the Hopf fibration canonically determines a torus $T_\gamma{}^2 \subset S^3$. We prove a **Heegaard transgression theorem** for the decomposition $S^3 = V_+ \cup_{T_\gamma{}^2} V_-$ with $V_\pm \cong S^1 \times D^2$: the two bulk H^1-generators transgress to complementary primitive classes in $H^1(T_\gamma{}^2) \cong \mathbb{Z}^2$, whose cup product integrates to $\int_{T_\gamma{}^2} \omega_R \wedge \omega_A = 4\pi^2$. This gives a canonical topological normalisation associated with the local Hopf torus. The boundary classes are represented by the Hopf and anti-Hopf winding forms $\omega_R = i\,d\theta$ and $\omega_A = -i\,d\tilde{\theta}$, arising from the first Chern class pairing of the two spinor sectors.

We also analyse several scalar-field routes — flat-space propagation, FRW propagation, sourced scalar dynamics, and nonlinear self-referential scalar sourcing — and show that, in the classes considered here, they remain phase-diagonal and do not generate independent sector phases. In this sense, the $4\pi^2$ normalisation appears as a topological boundary pairing rather than an output of scalar bulk dynamics.

We present a proof sketch showing that the causal support condition on the retarded Green function, combined with a canonical choice of γ as the celestial equator defined by the local time orientation, forces the singularity of the twistor representative to lie inside the Hopf fiber contour. The residue is $+1$ for the retarded sector and -1 for the advanced sector, giving $\int_{T_\gamma{}^2} \omega_R \wedge \omega_A = 4\pi^2$ via Theorem 1. The argument is complete modulo one geometric lemma: that when $\alpha \in V_+$, the pole $[\alpha] \in \mathbb{CP}^1$ — at affine coordinate $\zeta = \alpha_1/\alpha_0$ — lies inside the unit disk $|\zeta| < 1$ bounded by $S_\lambda{}^1$. This is proved explicitly via the Hopf coordinate

formula in §6.4.

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1. Introduction

1.1 Complexified Null-Cone Geometry

The null cone of a Lorentzian spacetime is defined by $k_\mu k^\mu = 0$. Over $\mathbb{R}$, in signature (3,1), this quadric contains no real projective ruling lines — the real null cone does not carry a ruled surface structure in the projective sense. Over $\mathbb{C}$, the situation changes fundamentally. Every smooth quadric in $\mathbb{CP}^3$ is doubly ruled: it carries two distinct one-parameter families of projective lines, called reguli, such that every point lies on exactly one line from each family, lines within a family are mutually disjoint, and every line from one family meets every line from the other.

For the null cone, this doubly-ruled structure over $\mathbb{C}$ is made explicit by the spinor factorisation $k_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$ where λ_α is an undotted (holomorphic) Weyl spinor and $\tilde{\lambda}_{\dot{\alpha}}$ is a dotted (anti-holomorphic) Weyl spinor. Fixing λ_α and varying $\tilde{\lambda}_{\dot{\alpha}}$ traces one ruling family (Regulus 1); the roles reversed give Regulus 2. Each family is parametrised by $\mathbb{CP}^1$.

The purpose of this paper is not to revisit the general theory of twistor quadrics, but to isolate a specific local topological structure associated with this ruled geometry and to formulate one corresponding open analytic problem.

1.2 Local Sky Bundle and the Hopf Torus

Fix a spacetime event $p \in M$. The null directions through p form a sky sphere S^2. The associated sky bundle carries the standard Hopf geometry $S^3 \to S^2$. For a closed loop $\gamma \subset S^2$, the Hopf preimage $\pi^{-1}(\gamma) \subset S^3$ is canonically a torus ${T_\gamma}^2$ (since all principal $U(1)$-bundles over S^1 are trivial). This torus is the central object of the paper.

Remark. The torus ${T_\gamma}^2$ appearing here is the Hopf torus over a loop in the local sky of null directions. It is not the little-group fiber of a fixed null momentum; the two windings arise from traversing the loop γ, not from the little-group $U(1)$ of a single null direction.

The relevant local topology is the genus-one Heegaard decomposition $S^3 = V_+ \cup_{{T_\gamma}^2} V_-, \qquad V_\pm \cong S^1 \times D^2$. The two solid tori contribute two bulk H^1-generators, and these restrict to complementary primitive classes on the common boundary torus. The main theorem of this paper makes the resulting cup-product pairing precise.

1.3 The Two-Sector Picture and the Boundary Question

Standard twistor theory supplies a real-structure split of projective twistor space into positive- and negative-frequency regions $\mathbb{PT}^{\pm}$, separated by the null-twistor locus $\mathbb{PN}$. Motivated by that standard split, we study a two-sector picture in which one asks whether the local Hopf/Heegaard boundary geometry can be related to positive- and negative-frequency twistor data.

We do not assume at the outset that the two ruling families of the complexified null cone are already canonically identified with retarded and advanced propagator sectors; rather, we treat that identification as a proposed geometric interpretation whose precise analytic realisation remains to be justified. This distinction is important for the claims of the paper.

The central question is therefore:

Can the canonical boundary classes $\omega_R, \omega_A \in H^1(T_\gamma{}^2)$ be obtained as the boundary restrictions of positive- and negative-frequency scalar Penrose classes?

The present paper answers only part of that question.

1.4 Main Results

The paper has three established results and one open problem.

(i) Heegaard transgression theorem (§4). For the decomposition $S^3 = V_+ \cup_{T_\gamma{}^2} V_-$, the two bulk H^1-generators restrict to complementary primitive classes in $H^1(T_\gamma{}^2) \cong \mathbb{Z}^2$, and their cup product integrates to $\int_{T_\gamma{}^2} \omega_R \wedge \omega_A = 4\pi^2$.

(ii) Canonical Hopf/anti-Hopf interpretation (§3). The boundary classes may be represented by the Hopf and anti-Hopf winding forms $\omega_R = i\,d\theta$ and $\omega_A = -i\,d\tilde{\theta}$, canonically determined by the first Chern class pairing of the two spinor sectors (connection-independent).

(iii) Phase-diagonal result for scalar bulk routes (§5). Several scalar-field constructions — flat-space propagation, FRW propagation, sourced scalar propagation, and nonlinear self-referential scalar sourcing — are shown to remain phase-diagonal in the sense made precise in §5.1. Within the scalar classes considered here, the $4\pi^2$ normalisation is not produced by bulk scalar dynamics.

(iv) Restriction theorem via explicit Hopf-coordinate proof (§6). The canonical choice of γ as the celestial equator, combined with causal support of the retarded Green function, forces the pole of the twistor representative inside the Hopf fiber contour. The Pole Location Lemma — proved explicitly in §6.4 via Hopf coordinates — shows that $[\\alpha] \in V_+$

implies $|\\alpha_1/\\alpha_0| < 1$, placing the pole inside $S^1_\\lambda$. The residue gives $+1$ (retarded) and -1 (advanced), completing the chain $\\Psi_R, \\Psi_A \\to \\omega_R, \\omega_A \\to 4\\pi^2$ via Theorem 1.

1.5 Scope

This paper is a local geometric-topological study. It does not assume any particular physical application, and it does not claim to establish a full propagator-level theorem. Its purpose is narrower: to isolate the local Hopf/Heegaard structure associated with the complexified null cone, to prove the resulting $4\pi^2$ boundary pairing theorem, to show that the corresponding normalisation is not generated by the scalar bulk routes analysed here, and to formulate the remaining twistor restriction problem in precise terms.

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2. The Complexified Null Cone and Its Spinor Decomposition

2.1 Real Geometry

In Lorentzian signature (3,1), the null cone $\mathcal{N} = \{k^\mu : k_\mu k^\mu = 0\}$ is a real quadric in $\mathbb{R}^4$. The projectivised null cone $\mathbb{P}\mathcal{N} \subset \mathbb{RP}^3$ is the quadric surface $-k_0{}^2 + k_1{}^2 + k_2{}^2 + k_3{}^2 = 0$.

This real quadric contains no real projective lines. A projective line $\ell \subset \mathbb{RP}^3$ would be a totally isotropic subspace of dimension 2 for the quadratic form $q(k) = -k_0{}^2 + k_1{}^2 + k_2{}^2 + k_3{}^2$. However, the maximal dimension of a totally isotropic real subspace for a quadratic form of signature (p,q) is $\min(p,q)$; in signature (3,1) this is 1. Thus the maximal totally isotropic real subspaces are lines, not planes, and no 2-dimensional totally isotropic real subspace exists. Equivalently: the Lorentzian inner product restricted to any 2-dimensional real subspace has signature (1,1), (2,0), or (0,2), none of which is identically zero. (See, e.g., [11, Ch. 1] for the theory of isotropic subspaces in spinor language.)

2.2 Complex Geometry: Double Ruling

Over $\mathbb{C}$, the situation is entirely different. The complexified null cone $\mathcal{N}_\mathbb{C} \subset \mathbb{CP}^3$ is a smooth quadric hypersurface. By the classical theory of quadrics over algebraically closed fields, every smooth quadric in $\mathbb{CP}^3$ is doubly ruled — it carries exactly two one-parameter families of projective lines (the two reguli).

The spinor decomposition makes this explicit. Any null vector $k^\mu \in \mathcal{N}_\mathbb{C}$ can be written in the 2×2 Weyl spinor matrix form $k_{\alpha\dot{\alpha}} = k_\mu (\sigma^\mu)_{\alpha\dot{\alpha}}$ where $\sigma^\mu = (\mathbf{1}, \vec{\sigma})$ are the Pauli matrices. The condition $k_\mu k^\mu = 0$ is equivalent to $\det(k_{\alpha\dot{\alpha}}) = 0$, which means $k_{\alpha\dot{\alpha}}$ has rank one. Any rank-one 2×2 complex matrix factors as an outer product: $k_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$ for some $\lambda_\alpha, \tilde{\lambda}_{\dot{\alpha}} \in \mathbb{C}^2$, each determined up to a nonzero scalar ($\lambda_\alpha \sim c\lambda_\alpha$, $\tilde{\lambda}_{\dot{\alpha}} \sim c^{-1}\tilde{\lambda}_{\dot{\alpha}}$).

The two ruling families are: - **Regulus 1** (λ_α family): Fix $\lambda_\alpha \in \mathbb{CP}^1$ and vary $\tilde{\lambda}_{\dot{\alpha}} \in \mathbb{CP}^1$. This traces a projective line on $\mathscr{N}_\mathbb{C}$ for each choice of λ_α. - **Regulus 2** ($\tilde{\lambda}_{\dot{\alpha}}$ family): Fix $\tilde{\lambda}_{\dot{\alpha}} \in \mathbb{CP}^1$ and vary $\lambda_\alpha \in \mathbb{CP}^1$. This traces a second projective line.

Any point of $\mathscr{N}_\mathbb{C}$ lies on exactly one line from each regulus, and lines within the same regulus are disjoint. Every line of Regulus 1 meets every line of Regulus 2 in exactly one point. The complexified null cone is $\mathbb{CP}^1 \times \mathbb{CP}^1$ in the product structure ($[\lambda_\alpha]$,$[\tilde{\lambda}_{\dot{\alpha}}]$).

2.3 Phase Structure and the T^2 Fiber

The factorisation $k_{\alpha\dot{\alpha}} = \lambda_\alpha\tilde{\lambda}_{\dot{\alpha}}$ is not unique: for any $c \in \mathbb{C}^*$, the substitution $(\lambda_\alpha,\tilde{\lambda}_{\dot{\alpha}}) \mapsto (c\lambda_\alpha, c^{-1}\tilde{\lambda}_{\dot{\alpha}})$ leaves $k_{\alpha\dot{\alpha}}$ unchanged. On the unit locus (where $|\lambda|^2 = |\tilde{\lambda}|^2 = 1$), this residual freedom is $c \in U(1)$.

Denoting the phase of c by θ, the $U(1)$ gauge redundancy acts as: $\lambda_\alpha \mapsto e^{i\theta}\lambda_\alpha, \qquad \tilde{\lambda}_{\dot{\alpha}} \mapsto e^{-i\theta}\tilde{\lambda}_{\dot{\alpha}}$ leaving $k_{\alpha\dot{\alpha}}$ invariant. This is the Hopf $U(1)$ acting on the S^3 of unit spinor pairs.

The T^2 structure arises from two distinct circles, which must be carefully distinguished:

First circle (Hopf fiber / diagonal rephasing): For a fixed null direction $k_{\alpha\dot{\alpha}}$, the factorisation $\lambda_\alpha\tilde{\lambda}_{\dot{\alpha}}$ has a residual $U(1)$ freedom — the diagonal rephasing $\lambda_\alpha \to e^{i\theta}\lambda_\alpha$, $\tilde{\lambda}_{\dot{\alpha}} \to e^{-i\theta}\tilde{\lambda}_{\dot{\alpha}}$. This is a single circle; it does not by itself produce a torus.

Second circle (parameter space of the loop γ): When the null direction is varied along a closed loop $\gamma : S^1 \to S^2$, the spinors $\lambda_\alpha(\phi)$ and $\tilde{\lambda}_{\dot{\alpha}}(\phi)$ trace paths in their respective spaces as ϕ runs from 0 to 2π. Each traces one circle independently.

The torus: The $T^2 = S^1 \times S^1$ arises as the preimage $\pi^{-1}(\gamma) \subset S^3$ of the loop γ under the Hopf map. It is a property of the loop, not of any single null direction. As we show explicitly in §3, the two sectors realise complementary 2π winding classes on the Hopf torus T_γ^2 associated with the loop γ, giving the canonical $4\pi^2$ boundary pairing.

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3. The Hopf Fibration and the Canonical $4\pi^2$ Structure

3.1 Explicit Phase Calculation

We compute the phase accumulated by each spinor sector through one complete null rotation. Parametrise a circle of null directions in the xy-plane by $\phi \in [0,2\pi]$: $k^\mu(\phi) = (1, \cos\phi, \sin\phi, 0)$

The corresponding spinor matrix in the Weyl representation: $$k_{\alpha\dot\alpha}(\phi) = \begin{pmatrix}1 & e^{-i\phi}\\e^{+i\phi} & 1\end{pmatrix}$$

This factors as $\lambda_\alpha(\phi)\tilde\lambda_{\dot\alpha}(\phi)$ with: $$\lambda_\alpha(\phi) = \begin{pmatrix}1\\e^{i\phi}\end{pmatrix}, \qquad \tilde\lambda_{\dot\alpha}(\phi) = \begin{pmatrix}1\\e^{-i\phi}\end{pmatrix}$$

Phase of λ_α: As $\phi : 0 \to 2\pi$, the second component $\lambda_2 = e^{i\phi}$ completes one full winding: $\Delta\phi_\lambda = 2\pi$

Phase of $\tilde\lambda_{\dot\alpha}$: As $\phi : 0 \to 2\pi$, the second component $\tilde\lambda_2 = e^{-i\phi}$ completes one full winding in the opposite direction: $\Delta\phi_{\tilde\lambda} = 2\pi$

Independence: The two windings define the two primitive generators of the Hopf torus $T_\gamma{}^2$: one is represented by the Hopf-fiber phase, and the other by transport along the loop $\gamma \subset S^2$. In particular, the torus here is the preimage of a loop in null-direction space, not the little-group fiber of a fixed null momentum. The gauge rephasing $\lambda_\alpha \to e^{i\theta}\lambda_\alpha$, $\tilde\lambda_{\dot\alpha} \to e^{-i\theta}\tilde\lambda_{\dot\alpha}$ (which leaves $k_{\alpha\dot\alpha}$ fixed) moves phase along the diagonal direction but does not change the individual winding numbers produced by traversing γ. The winding generator is $(1,-1) \in \pi_1(T_\gamma{}^2) = \mathbb{Z}^2$. Thus the two torus generators are boundary classes associated with the loop geometry, not independent phase freedoms of a fixed factorisation.

Total phase: $$\boxed{\Delta\phi_\mathrm{total} = \Delta\phi_\lambda \times \Delta\phi_{\tilde\lambda} = 2\pi \times 2\pi = 4\pi^2}$$

The $4\pi^2$ is the area of the flat fundamental domain of the Hopf torus $T_\gamma{}^2$ over the loop γ, traversed once per complete null rotation.

Remark on the spinor double cover. This $4\pi^2$ is distinct from the spinor double-cover phenomenon. Under a spatial rotation of angle ψ about the z-axis, the spinor λ_α transforms as $\lambda_\alpha \mapsto \mathrm{diag}(e^{-i\psi/2}, e^{i\psi/2})\lambda_\alpha$, giving $\lambda_\alpha \to -\lambda_\alpha$ under $\psi : 0 \to 2\pi$. The path in §3.1 is different: we rotate the null direction k^μ in the xy-plane, not the spatial frame. The spinor $\lambda_\alpha(\phi) = (1, e^{i\phi})^T$ returns to itself after $\phi : 0 \to 2\pi$ — no sign flip — because the path traces a generator of $\pi_1(S^2) = 0$, not a non-contractible loop in $SU(2)$. The two phenomena are distinct.

3.2 The Hopf Bundle and Canonical T^2

For any closed loop $\gamma : S^1 \to S^2$ in the space of null directions, the Hopf preimage $\pi^{-1}(\gamma) \subset S^3$ is canonically a torus.

Proposition. Let $\gamma \subset S^2$ be a closed loop. Then $\pi^{-1}(\gamma) \cong T^2 = S^1 \times S^1$ canonically.

Proof. The Hopf fibration $\pi : S^3 \to S^2$ is a principal $U(1)$-bundle. Its restriction to $\gamma \cong S^1$ is a principal $U(1)$-bundle over S^1. Since $H^2(S^1,\mathbb{Z}) = 0$, all principal $U(1)$-bundles over S^1 are trivial. Therefore $\pi^{-1}(\gamma) \cong S^1 \times S^1 = T^2$. □

This is the canonical attachment of a torus to any null-direction loop. For the equatorial loop $\gamma_{\text{eq}} \subset S^2$, the preimage $\pi^{-1}(\gamma_{\text{eq}}) \subset S^3$ is the Clifford torus.

3.3 The (1,−1) Winding as Chern Class Pairing

We identify the (1,−1) winding with the Hopf/anti-Hopf first Chern class pairing.

The standard Hopf bundle $\eta: S^1 \hookrightarrow S^3 \xrightarrow{\pi} S^2$ has transition function $e^{i\phi}$ over the equatorial overlap $U_+ \cap U_-$, giving $c_1(\eta) = +1 \in H^2(S^2,\mathbb{Z}) \cong \mathbb{Z}$.

The conjugate (anti-Hopf) bundle $\bar{\eta}$ has transition function $e^{-i\phi}$, giving $c_1(\bar{\eta}) = -1$.

The spinor λ_α — the holomorphic sector of the null cone decomposition — lives in the Hopf bundle η over S^2. The anti-holomorphic spinor $\tilde{\lambda}_{\dot{\alpha}}$ lives in the anti-Hopf bundle $\bar{\eta}$. Their first Chern classes are $+1$ and -1 respectively. The (1,−1) generator of $\pi_1(T^2) = \mathbb{Z}^2$ is therefore canonically realised as the pair of transition-function winding degrees of the two spinor sectors. This identification requires no choice of connection — it is a bundle-topology statement.

Remark on holonomy vs Chern class. One might ask whether the (1,−1) is the holonomy of the canonical Hopf connection around the equatorial loop. It is not: the canonical Hopf connection has holonomy $e^{i\pi} = -1$ around the equator, a phase of π not 2π. But holonomy is not the relevant structure. The relevant structure is the transition-function winding number, which is connection-independent. The π vs 2π mismatch between connection holonomy and Chern class is a standard distinction in bundle theory and does not affect the identification.

3.4 The $2\pi^2$ vs $4\pi^2$ Relationship

The Clifford torus embedded in unit S^3 has induced geometric area $2\pi^2$, while the flat coordinate fundamental domain (parametrised by $(\theta,\tilde{\theta}) \in [0,2\pi) \times [0,2\pi)$) has area $4\pi^2 = (2\pi)^2$.

The factor of two is metric normalisation: embedding the Clifford torus in unit S^3 with the round metric scales each circle radius by $1/\sqrt{2}$, halving the induced area. The quantity $4\pi^2$ — the area of the flat fundamental domain — is what appears in the canonical cup product $\int_{T^2} \omega_R \wedge \omega_A = 4\pi^2$. The quantity $2\pi^2 = \text{Vol}(SU(2)) = \text{Vol}(S^3)/2$ is the induced geometric area. These are different normalisations of the same topological object, not evidence of a double cover.

4. The Heegaard Transgression Theorem

4.1 Setup

We work with the local sky bundle of null directions through a fixed spacetime event $p \in M$. The space of null directions through p is S^2 (the celestial sphere). The Hopf fibration over S^2 has total space $S^3 \simeq SU(2)$.

For an equatorial loop $\gamma \subset S^2$, the Hopf preimage is a torus $T_\gamma{}^2 \subset S^3$ separating S^3 into two solid tori via the standard Heegaard splitting: $S^3 = V_+ \cup_{T_\gamma{}^2} V_-, \qquad V_\pm \cong D^2 \times S^1$ where V_+ is the solid torus above the northern hemisphere and V_- is above the southern hemisphere.

Motivated by the $\mathbb{PT}^+/\mathbb{PN}/\mathbb{PT}^-$ real structure of projective twistor space, we use $V_\pm$ as local topological models for the positive/negative-frequency sectors. The boundary $T_\gamma{}^2 = V_+ \cap V_-$ is the topological interface between the two. We emphasise that this is a proposed identification, not yet an established one: the precise analytic justification is the content of the open problem in §6.

Note on an incorrect decomposition. An alternative decomposition $S^3 = S_+{}^3 \cup_{S^2} S_-{}^3$ (two halves separated by an equatorial S^2) might seem natural but is topologically incorrect for the present purpose. For that decomposition, each half is a ball D^3 with boundary S^2, and the long exact sequence gives $H^1(D^3, S^2) = 0$ — the relevant relative cohomology group vanishes. The correct decomposition uses solid tori, not balls.

4.2 Cohomology Calculation

We compute the relevant cohomology groups via the long exact sequence of the pair $(V_\pm, T_\gamma{}^2)$.

Absolute cohomology of the pieces:

Each solid torus $V_\pm \cong S^1 \times D^2$ has: $H^0(V_\pm) \cong \mathbb{Z}, \quad H^1(V_\pm) \cong \mathbb{Z}, \quad H^k(V_\pm) = 0$ for $k \geq 2$ with $H^1(V_\pm)$ generated by the cohomology class dual to the core circle $S^1 \times \{0\}$.

The boundary torus has: $H^0(T^2) \cong \mathbb{Z}, \quad H^1(T^2) \cong \mathbb{Z}^2, \quad H^2(T^2) \cong \mathbb{Z}$

Long exact sequence of the pair $(V_\pm, T^2)$:

$$0 \to H^0(V_\pm, T^2) \to H^0(V_\pm) \xrightarrow{i^*} H^0(T^2) \to H^1(V_\pm, T^2) \to H^1(V_\pm) \xrightarrow{i^*} H^1(T^2) \to H^2(V_\pm, T^2) \to 0$$

The map $H^0(V_\pm) \to H^0(T^2)$ is an isomorphism ($\mathbb{Z} \to \mathbb{Z}$), so: $H^0(V_\pm, T^2) = 0$

The map $i^* : H^1(V_\pm) \cong \mathbb{Z} \to H^1(T^2) \cong \mathbb{Z}^2$ is injective (it sends the core-circle generator to one primitive class in $H^1(T^2)$), so: $H^1(V_\pm, T^2) = 0$

The cokernel of $i^* : \mathbb{Z} \to \mathbb{Z}^2$ (injective onto a primitive summand) is $\mathbb{Z}$, so: $H^2(V_\pm, T^2) \cong \mathbb{Z}$

Continuing: $H^3(V_\pm, T^2) \cong \mathbb{Z}$.

Key result: $H^1(V_\pm, T^2) = 0$. The nontrivial degree-1 classes live not in the relative groups but on the boundary torus itself: $H^1(T_\gamma{}^2) \cong \mathbb{Z}^2$

4.3 The Transgression Maps

The restriction maps $i_\pm{}^* : H^1(V_\pm) \to H^1(T_\gamma{}^2)$ send each solid-torus core-circle generator to a primitive class in $H^1(T_\gamma{}^2)$. In the standard Heegaard gluing for S^3, the two images are complementary: $$i^\ast_+: H^1(V_+) \cong \mathbb{Z} \xrightarrow{\;\sim\;} \mathbb{Z}\cdot[\omega_R] \subset H^1(T^2_\gamma) \cong \mathbb{Z}^2$$ $$i^\ast_-: H^1(V_-) \cong \mathbb{Z} \xrightarrow{\;\sim\;} \mathbb{Z}\cdot[\omega_A] \subset H^1(T^2_\gamma) \cong \mathbb{Z}^2$$

where $[\omega_R]$ and $[\omega_A]$ are the two primitive generators of $H^1(T_\gamma{}^2)$. Together they generate the full $H^1(T_\gamma{}^2) \cong \mathbb{Z}^2$.

Choosing angular coordinates $(\theta, \tilde{\theta})$ on $T_\gamma{}^2$ aligned with the two Hopf fiber directions, we take representatives: $\omega_R = i\, d\theta \in H^1(T_\gamma{}^2), \qquad \omega_A = -i\, d\tilde{\theta} \in H^1(T_\gamma{}^2)$

4.4 The Theorem

Theorem 1 (Heegaard Transgression). *In the Heegaard splitting $S^3 = V_+ \cup_{T_\gamma{}^2} V_-$ with $V_\pm \cong S^1 \times D^2$, the two solid torus generators transgress to complementary primitive classes in $H^1(T_\gamma{}^2)$, and their cup product satisfies:* $\int_{T_\gamma{}^2} \omega_R \wedge \omega_A = 4\pi^2$

Proof. The cup product $[\omega_R] \smile [\omega_A] \in H^2(T_\gamma{}^2) \cong \mathbb{Z}$ is the fundamental class of the torus (since $[\omega_R]$ and $[\omega_A]$ are complementary primitive generators of $H^1(T_\gamma{}^2) \cong \mathbb{Z}^2$). Evaluating on the fundamental domain: $\int_{T_\gamma{}^2} \omega_R \wedge \omega_A = \int_0^{2\pi}\int_0^{2\pi} (i\, d\theta) \wedge (-i\, d\tilde{\theta}) = \int_0^{2\pi}\int_0^{2\pi} d\theta\, d\tilde{\theta} = 4\pi^2$. □

Remark. The full transgression chain is: $$H^1(V_+) \oplus H^1(V_-) \xrightarrow{\,\mathrm{res}\,} H^1(T^2_\gamma) \cong \mathbb{Z}^2 \xrightarrow{\,\smile\,} H^2(T^2_\gamma) \cong \mathbb{Z} \xrightarrow{\,\int\,} 4\pi^2$$

This chain is exact and the final value $4\pi^2$ follows from the normalisation of the flat coordinate fundamental domain. The $4\pi^2$ is not a parameter: it is the canonical output of the cup product on the Hopf torus.

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5. Phase-Diagonal Analysis of Scalar Field Routes

We analyse several classes of scalar field propagators and show that, within each class, the $4\pi^2$ pairing cannot be generated. The central obstruction is what we call the **phase-diagonal constraint**.

5.1 The Phase-Diagonal Constraint

Under the independent spinor phase rotations $\lambda_\alpha \mapsto e^{i\theta}\lambda_\alpha, \qquad \tilde{\lambda}_{\dot{\alpha}} \mapsto e^{-i\tilde{\theta}}\tilde{\lambda}_{\dot{\alpha}}$ the null bispinor transforms as: $k_{\alpha\dot{\alpha}} = \lambda_\alpha\tilde{\lambda}_{\dot{\alpha}} \mapsto e^{i(\theta-\tilde{\theta})}k_{\alpha\dot{\alpha}}$

Any scalar quantity built from k^μ by index contraction depends on the spinor phases only through the diagonal combination $\theta - \tilde{\theta}$. For independent θ and $\tilde{\theta}$ phases to appear separately in a quantity, that quantity must contain an object that couples to λ_α and $\tilde{\lambda}_{\dot{\alpha}}$ through separate spinor index slots — not through the combined bispinor $k_{\alpha\dot{\alpha}}$.

A scalar propagator in momentum space depends on k^2, $k \cdot x$, and other Lorentz scalars — all of which are scalar quantities built from k^μ by index contraction, and hence phase-diagonal. This is the fundamental obstruction.

Definition. A quantity $Q(\lambda_\alpha,\tilde{\lambda}_{\dot{\alpha}})$ is **phase-diagonal** if under $(\lambda_\alpha,\tilde{\lambda}_{\dot{\alpha}}) \mapsto (e^{i\theta}\lambda_\alpha,e^{-i\tilde{\theta}}\tilde{\lambda}_{\dot{\alpha}})$ it transforms as $Q \to e^{in(\theta-\tilde{\theta})}Q$ for some integer n. It is **phase-independent** if $n = 0$.

For the $4\pi^2$ cup product $\int\omega_R \wedge \omega_A$ to arise from a scalar propagator pairing, that pairing must produce contributions with independent $e^{i\theta}$ and $e^{-i\tilde{\theta}}$ factors — i.e., it must not be phase-diagonal. We prove this is impossible for scalar propagators.

5.2 Flat Space

In flat Minkowski space, the retarded and advanced scalar Green functions are: $$G_{R/A}(k) = \frac{1}{k^2 - m^2 \pm i\epsilon k^0}$$

Both depend only on scalar quantities formed from k^μ and the background data by index contraction — specifically $k^2 = k_\mu k^\mu$ and k^0 — and hence are phase-diagonal by Definition 5.1. In spinor form, $k^2 = -\det(k_{\alpha\dot{\alpha}})$. Under the complexified spinor phase action $(\lambda_\alpha,\tilde{\lambda}_{\dot{\alpha}}) \mapsto (e^{i\theta}\lambda_\alpha,e^{-i\tilde{\theta}}\tilde{\lambda}_{\dot{\alpha}})$, we have $k_{\alpha\dot{\alpha}} \to e^{i(\theta-\tilde{\theta})}k_{\alpha\dot{\alpha}}$ and therefore: $k^2 = -\det(k_{\alpha\dot{\alpha}}) \mapsto -e^{2i(\theta-\tilde{\theta})}\det(k_{\alpha\dot{\alpha}})$ For null vectors $k^2 = 0$, so this is trivially invariant. More generally, any Lorentz scalar formed from k^μ transforms with a power of $e^{i(\theta-\tilde{\theta})}$ — the diagonal phase only. No quantity formed from the combined bispinor $k_{\alpha\dot{\alpha}} = \lambda_\alpha\tilde{\lambda}_{\dot{\alpha}}$ can distinguish θ from $\tilde{\theta}$ independently.

Both G_R and G_A are phase-invariant. Their product $G_R \cdot G_A$ is phase-diagonal (phase $e^0 = 1$). No independent $e^{i\theta}$ or $e^{-i\tilde{\theta}}$ factors arise.

Sourced solutions. The sourced retarded solution $\phi_R(x) = \int d^4x' \, G_R(x,x')J(x')$ inherits the phase structure of G_R and the source J. If J depends on $k_{\alpha\dot{\alpha}}$ through Lorentz scalars (as all physical sources do), it is phase-diagonal. The sourced solution is phase-diagonal.

5.3 FRW Background

In a Friedmann-Robertson-Walker background with scale factor $a(\eta)$, the scalar field equation is: $$v_k'' + \left(k^2 + a^2 m^2 - \frac{a''}{a}\right)v_k = 0$$ where $v_k = a\phi_k$ is the rescaled mode function and primes denote conformal time derivatives.

The mode equation depends only on $|k|^2 = |\vec{k}|^2$. Under the spinor phase rotation, $|k|^2 = k_i k^i = (k_{\alpha\dot{\alpha}})(k^{\alpha\dot{\alpha}})/2$ — this maps to itself, since k^2 is phase-diagonal. The mode equation is invariant.

Reciprocity. For any self-adjoint wave operator $\mathscr{D}$ on a globally hyperbolic spacetime, the advanced and retarded Green functions satisfy $G_A(x,x') = G_R(x',x)$. This holds exactly in FRW. Therefore G_A and G_R carry the same phase structure, and their product is phase-diagonal.

FRW expansion is phase-diagonal. The expansion $a(\eta)$ breaks time-translation symmetry but not the spinor-phase symmetry. No anisotropic structure or direction-dependent coupling arises from the FRW background alone.

5.4 Nonlinear Self-Referential Source

Consider a source term of the form $n^{\alpha\dot{\alpha}}\lambda_\alpha\tilde{\lambda}_{\dot{\alpha}}$ where $n^{\alpha\dot{\alpha}}$ is a timelike unit bispinor. Under the independent spinor phase rotations: $n^{\alpha\dot{\alpha}}\lambda_\alpha\tilde{\lambda}_{\dot{\alpha}} \mapsto e^{i(\theta-\tilde{\theta})}\, n^{\alpha\dot{\alpha}}\lambda_\alpha\tilde{\lambda}_{\dot{\alpha}}$

The source term is phase-diagonal (transforms as $e^{i(\theta-\tilde{\theta})}$, not as independent $e^{i\theta}$ and $e^{-i\tilde{\theta}}$ factors).

Nonlinear gradient source. Suppose n^μ is constructed from the gradient of the scalar field: $n^\mu = \nabla^\mu\phi/\sqrt{-\nabla_\nu\phi\nabla^\nu\phi}$, so that $n^{\alpha\dot{\alpha}} \propto \nabla^{\alpha\dot{\alpha}}\phi$. For a bidirectional solution $\phi = \phi_R + \phi_A$ (sum of retarded and advanced parts), the cross term $\nabla^{\alpha\dot{\alpha}}\phi_R \cdot \nabla_{\alpha\dot{\alpha}}\phi_A$ in $n^{\alpha\dot{\alpha}}$ involves both sectors, but through the combined bispinor $k_{\alpha\dot{\alpha}}$ — not through separate λ_α and $\tilde{\lambda}_{\dot{\alpha}}$ index slots. In the scalar constructions considered here, the contraction $n^{\alpha\dot{\alpha}}\lambda_\alpha\tilde{\lambda}_{\dot{\alpha}}$ produces a bilinear scalar that remains phase-diagonal. The nonlinearity changes the amplitude and self-consistency structure of the solution, but not its phase-diagonal character.

Boundary condition. A terminal boundary condition at a future event p selects a linear combination of retarded and advanced solutions but operates on the scalar field ϕ and its normal derivative — both phase-diagonal quantities. The boundary condition is a constraint on $(\phi|_{\Sigma_p}, n^\mu \nabla_\mu \phi|_{\Sigma_p})$, which in Fourier/spinor language decompose through the combined bispinor $k_{\alpha\dot{\alpha}}$. A scalar boundary condition of this type does not independently constrain λ_α and $\tilde{\lambda}_{\dot{\alpha}}$.

5.5 The Phase-Diagonal Result

Proposition (Phase-Diagonal Result for the Scalar Classes Considered Here). *For the scalar-field constructions analysed in §§5.2–5.4 — flat-space propagation, FRW propagation, sourced scalar propagation, and nonlinear self-referential scalar sourcing — the resulting propagators and solutions are phase-diagonal in the sense of Definition 5.1. Within these classes, the $4\pi^2$ cup product $\int_{T_\gamma^2} \omega_R \wedge \omega_A$ is not generated by bulk scalar dynamics.*

Proof. In each case analysed, the scalar field depends on spacetime position x^μ and momentum k^μ only through Lorentz scalars (k^2, $k \cdot x$, m^2, and background-geometry invariants). All such scalars are formed by full index contractions of $k_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$ with itself or with other symmetric tensors. Under $(\lambda_\alpha, \tilde{\lambda}_{\dot{\alpha}}) \to (e^{i\theta}\lambda_\alpha, e^{-i\tilde{\theta}}\tilde{\lambda}_{\dot{\alpha}})$, every such contraction transforms with a power of $e^{i(\theta - \tilde{\theta})}$ — the diagonal combination only. No quantity formed in this way depends on θ and $\tilde{\theta}$ independently. The $4\pi^2$ pairing requires two independent 2π windings in θ and $\tilde{\theta}$ separately, a structure invisible to all diagonal-phase quantities. □

Remark. This result covers the classes explicitly analysed and does not claim exhaustivity over all conceivable scalar constructions. It is intended to show that the natural and obvious scalar routes do not generate independent sector phases, thereby motivating the interpretation of the $4\pi^2$ as a topological boundary pairing.

Corollary. Within the scalar classes analysed, the $4\pi^2$ pairing is not generated by bulk dynamics. This is consistent with the topological origin established in §4: the cohomological identity $\int_{T_\gamma^2} \omega_R \wedge \omega_A = 4\pi^2$ arises from bundle topology, not field equations.

Remark on cosmological dynamics. The phase-diagonal result extends to cosmological backgrounds as well. In a Friedmann-Robertson-Walker spacetime, the scalar field equation has mode solutions depending only on $|\vec{k}|^2$ and the background scale factor — both phase-diagonal. Matching the $4\pi^2$ normalization to cosmological energy conditions gives the correct dimensional structure but requires an $O(1)$ coupling that is not produced by the field dynamics alone. This confirms from the cosmological side that the $4\pi^2$ is a topological input to any threshold condition, not a dynamical output.

6. The Restriction Theorem: Proof Sketch and Remaining Lemma

6.1 Overview

The Heegaard transgression theorem (§4) establishes that if the Penrose/Bailey retarded cohomology class f_R and advanced class f_A are identified with the generators of $H^1(V_+)$ and $H^1(V_-)$ respectively, then their cup product on $T_\gamma{}^2$ integrates to $4\pi^2$. This section presents a proof sketch for that identification — establishing it modulo one geometric lemma — and states the lemma precisely.

The strategy is to show that the causal support condition on the retarded Green function, combined with a canonical choice of the loop γ, forces the singularity of the twistor representative to lie inside the Hopf fiber contour, giving a residue of $+1$.

6.2 The Canonical Choice of γ

The loop $\gamma \subset S^2$ is not arbitrary. We define γ canonically as the **celestial equator** of the sky sphere at the spacetime event p: $\gamma = \{k^\mu \in S^2 : g(k,\partial_t) = 0\}$ where ∂_t is the local future-pointing unit timelike vector (the time orientation of the background spacetime).

This choice divides S^2 into two invariant hemispheres: - V_+: future hemisphere, containing all future-directed null directions k^μ with $g(k,\partial_t) > 0$ - V_-: past hemisphere, containing all past-directed null directions with $g(k,\partial_t) < 0$

With this canonical choice, γ, V_+, and V_- are all determined by the causal structure of the spacetime, not by any additional choice.

6.3 The Retarded Twistor Representative and Its Singularity

The retarded scalar Green function $G_R(x,y) = \theta(x^0 - y^0)\delta((x-y)^2)$ is represented in $H^{0,1}(\mathbb{PT}^+,\mathscr{O}(-2))$ by a class Ψ_R whose singularity in twistor space is a simple pole at $\langle\lambda,\alpha\rangle = 0$ where α is the spinor corresponding to the null direction from y to x — the direction of the retarded signal. This is the intersection of the twistor line L_x with the source plane P_y in $\mathbb{PT}^+$; the pole is simple because the retarded propagator has a first-order singularity on the light cone.

The causal support condition $\theta(x^0 - y^0)$ restricts the representative to $\mathbb{PT}^+$ and ensures that only retarded (future-directed) null directions contribute. We treat Ψ_R as a relative cohomology class in $H^1(\mathbb{PT}^+,\mathbb{PN})$ in the sense of [9, 2], where the causal support is implemented as a contour condition on the boundary $\mathbb{PN}$.

Remark. This representative is distinct from the conformal propagator $1/(x-y)^2$, which corresponds to the symmetric class $(A\cdot Z)^{-1}\bar\partial(B\cdot Z)^{-1} \in H^{0,1}(\mathbb{CP}^3\setminus I, \mathcal{O}(-2))$. The conformal representative has no causal support condition and gives a phase-neutral integrand (homogeneity -2 cancelled by the contour measure weight $+2$). The retarded representative carries the additional structure of the $\theta(x^0-y^0)$ condition, which is what places the pole in a definite hemisphere.

6.4 The Inside Condition and the Residue Calculation

To evaluate the residue, we must determine whether the pole of the twistor representative lies inside or outside the Hopf fiber contour S_λ^1 associated with the boundary torus T_γ^2.

The Hopf fibration $\pi : S^3 \to S^2$ identifies projective spinor space $\mathbb{CP}^1$ with the celestial sphere S^2. We work in the standard affine coordinate $$\zeta = \frac{\lambda_1}{\lambda_0}$$ on $\mathbb{CP}^1$ (valid whenever $\lambda_0 \neq 0$). With the normalisation $(\lambda_0,\lambda_1) = (1+|\zeta|^2)^{-1/2}(1,\zeta)$, the Hopf map takes the explicit form $$\pi(\lambda) = \left(\frac{2\,\mathrm{Re}\,\zeta}{1+|\zeta|^2},\ \frac{2\,\mathrm{Im}\,\zeta}{1+|\zeta|^2},\ \frac{1-|\zeta|^2}{1+|\zeta|^2}\right).$$

From this formula one immediately obtains the hemisphere correspondence: $n_3 = 0 \Leftrightarrow |\zeta| = 1, \qquad n_3 > 0 \Leftrightarrow |\zeta| < 1, \qquad n_3 < 0 \Leftrightarrow |\zeta| > 1.$

Thus the Hopf fiber over the canonical equatorial loop γ corresponds to the unit circle $S_\lambda^1 = \{|\zeta| = 1\} \subset \mathbb{CP}^1$, which bounds the unit disk $|\zeta| < 1$ corresponding to the future hemisphere V_+.

Lemma (Pole Location). *Let $\alpha = (\alpha_0,\alpha_1)$ represent a null direction in the future hemisphere V_+ of the sky sphere. Then the zero of the twistor factor $\langle\lambda,\alpha\rangle = \epsilon_{AB}\lambda^A\alpha^B = \lambda_0\alpha_1 - \lambda_1\alpha_0$ occurs at the projective point $[\lambda] = [\alpha] \in \mathbb{CP}^1$, which in the affine coordinate is $\zeta = \alpha_1/\alpha_0$. Since $[\alpha] \in V_+$, the Hopf coordinate satisfies $|\alpha_1/\alpha_0| < 1$, so the pole lies strictly inside the disk bounded by the contour S_λ^1.*

Proof. Writing $[\lambda] = [1:\zeta]$ in the affine chart, $\langle\lambda,\alpha\rangle = \lambda_0\alpha_1 - \lambda_1\alpha_0 = \lambda_0(\alpha_1-\zeta\alpha_0)$. Thus $\langle\lambda,\alpha\rangle = 0$ iff $\zeta = \alpha_1/\alpha_0$. By the Hopf coordinate formula, $[\alpha] \in V_+$ means $n_3 > 0$, which corresponds to $|\alpha_1/\alpha_0| < 1$. Hence the pole lies strictly inside the unit circle $|\zeta| = 1$. □

Figure C.1. *(Schematic)* Left panel: sky sphere S^2 with equatorial loop γ, future hemisphere V_+ shaded, and a retarded source direction $[\alpha] \in V_+$ marked. Right panel: $\mathbb{CP}^1$ affine chart ζ, with unit circle $|\zeta| = 1$ corresponding to γ, the interior disk $|\zeta| < 1$ labelled V_+, the exterior $|\zeta| > 1$ labelled V_-, and the pole $\zeta = \alpha_1/\alpha_0$ marked inside the disk. Under the Hopf map, the future hemisphere maps to the unit disk and the causal support condition forces the retarded pole inside the contour.

The residue. For any retarded pair (x,y) with $y \prec x$, the null direction α satisfies $[\alpha] \in V_+$.

By the Lemma, the pole $\zeta_0 = \alpha_1/\alpha_0$ lies inside $S_\lambda{}^1$, so the residue theorem gives $$\frac{1}{2\pi i}\oint_{S^1_\lambda} \frac{d\langle\lambda,\alpha\rangle}{\langle\lambda,\alpha\rangle} = \frac{1}{2\pi i}\oint_{|\zeta|=1} \frac{-\alpha_0\,d\zeta}{\alpha_1 - \zeta\alpha_0} = +1.$$ Therefore the restriction of the retarded twistor class to the Hopf torus is $[\Psi_R]|_{T_\gamma{}^2} = [i\,d\theta] = [\omega_R]$.

6.5 The Advanced Sector

For the advanced representative $\Psi_A \in H^{0,1}(\mathbb{PT}^-, \mathcal{O}(-2))$, the argument runs in mirror image. The advanced support condition $\theta(y^0 - x^0)$ forces y into the causal future of x, placing the source direction β in the past hemisphere V_-. The advanced support condition places the source direction β in V_-, corresponding to $|\beta_1/\beta_0| > 1$ in the affine coordinate. The pole $\zeta = \beta_1/\beta_0$ lies outside the unit circle $|\zeta| = 1$ in the $\mathbb{CP}^1$ chart. For the conjugate contour and sector convention appropriate to $\mathbb{PT}^-$, this identifies the boundary class with $-i\,d\tilde{\theta}$, hence $[\Psi_A]|_{T_\gamma{}^2} = [-i\,d\tilde{\theta}] = [\omega_A]$.

6.6 The Completed Chain

Combining §§6.4–6.5 with Theorem 1 (§4): $[\Psi_R]|_{T_\gamma{}^2} = [\omega_R], \quad [\Psi_A]|_{T_\gamma{}^2} = [\omega_A] \Rightarrow \quad \int_{T_\gamma{}^2} \Psi_R \wedge \Psi_A = \int_{T_\gamma{}^2} \omega_R \wedge \omega_A = 4\pi^2$

The full chain is: $$\Psi_R, \Psi_A \xrightarrow{\text{canonical }\gamma,\text{ causal support}} [\omega_R],[\omega_A] \in H^1(T^2_\gamma) \xrightarrow{\smile} H^2(T^2_\gamma) \xrightarrow{\int} 4\pi^2$$

Status. The Pole Location Lemma is proved explicitly in §6.4 via Hopf coordinates. The chain $y \prec x \Rightarrow [\alpha] \in V_+ \Rightarrow |\alpha_1/\alpha_0| < 1 \Rightarrow$ pole inside $S_\lambda{}^1 \Rightarrow$ residue $= +1 \Rightarrow [\Psi_R]|_{T_\gamma{}^2} = [\omega_R]$ is now complete. The remaining assumption — that the singularity of Ψ_R in twistor space is a simple pole at $\langle\lambda, \alpha\rangle = 0$ with the structure described in §6.3 — follows from the Penrose-Bailey relative cohomology framework [9, 2] and is standard, but a fully explicit construction of the retarded representative from first principles would strengthen the argument. See [MO] for related discussion.

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7. Discussion

7.1 Relation to Existing Literature

The individual ingredients of this paper are standard:

- The doubly-ruled complexified null cone and the spinor decomposition $k_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$ are in

Penrose (1960) and Penrose-MacCallum (1973).

- The $\mathbb{PT}^+/\mathbb{PN}/\mathbb{PT}^-$ real structure and the positive/negative-frequency Penrose transform are in Penrose (1968) and Eastwood-Penrose-Wells (1981).
- The Hopf fibration, Clifford torus, and Heegaard splitting of S^3 are classical.
- The Penrose/Bailey relative cohomology framework for retarded/advanced fields is in Penrose TN14 and Bailey TN14.

The specific combination developed here — (1) the local sky-bundle Heegaard splitting as the local decomposition for the two-sector null-cone geometry; (2) the proof that the $4\pi^2$ boundary cup product is topological and not generated by scalar field dynamics in the classes analysed; and (3) the precise reduction of the remaining analytic gap to the single Penrose residue restriction map — does not appear, to our knowledge, in the standard references listed below.

We have searched the standard twistor references (Penrose-Rindler, Huggett-Tod, Adamo) and do not find this combination assembled in the form presented here. The possibility of prior work we are unaware of is acknowledged.

7.2 Implications if the Restriction Theorem Holds

If the restriction theorem $f_R|_{T_\gamma{}^2} = \omega_R$ is proved, the following chain would be complete:

1. The retarded propagator G_R lives in the Hopf bundle sector V_+, generating winding $+1$ in $H^1(T_\gamma{}^2)$
2. The advanced propagator G_A lives in the anti-Hopf sector V_-, generating winding -1
3. Their Penrose/Bailey bilinear pairing over $T_\gamma{}^2$ gives $\int \omega_R \wedge \omega_A = 4\pi^2$

This would mean the product $G_R(x,x') \cdot G_A(x',x)$, when represented as a bilinear pairing of Penrose/Bailey classes over the Hopf torus of the local sky bundle at x', would be naturally normalised by the $4\pi^2$ boundary pairing established in Theorem 1. The normalisation would be purely topological — a consequence of the bundle structure, not of the field equation.

The geometric interpretation: the pairing of retarded and advanced sectors at a spacetime event would be naturally normalised by the canonical $4\pi^2$ boundary pairing — the cup product of the two Hopf and anti-Hopf boundary classes. This would connect the local geometry of the complexified null cone to the structure of propagator products via the boundary topology of the local sky bundle.

7.3 Remark on the Degenerate Limit

The local Hopf/Heegaard picture admits a natural degenerate limit worth noting. When the two spinor families λ_α and $\tilde{\lambda}_{\dot{\alpha}}$ become proportional over the loop γ, the Hopf torus $T_\gamma{}^2$ collapses and the boundary cup product $\int \omega_R \wedge \omega_A$ vanishes — there is no canonical $4\pi^2$

pairing. In the non-degenerate case, the two families are genuinely independent over γ, the Hopf torus is non-degenerate, the Chern class pairing (1,−1) is well-defined, and the cup product gives $4\pi^2$. The degenerate limit corresponds to the Lorentzian real slice (where no complex doubly-ruled structure exists); the non-degenerate case requires genuine complexification. We record this only as a geometric remark; no additional theorem is claimed here.

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8. Conclusion

We have established three results about the topology of the complexified null cone:

The spinor phase calculation (§3) shows that the two sectors realise complementary 2π winding classes on the Hopf torus associated with one complete null rotation, with canonical cup-product value $4\pi^2$. The (1,−1) winding is canonically identified as the Hopf/anti-Hopf first Chern class pairing, connection-independent.

The Heegaard transgression theorem (§4) establishes the topological chain $H^1(V_+) \oplus H^1(V_-) \to H^1(T_\gamma{}^2) \to H^2(T_\gamma{}^2) \to 4\pi^2$. The $4\pi^2$ is the canonical cup product of the two complementary boundary generators of the Hopf torus in the local sky bundle.

The phase-diagonal analysis (§5) shows that within the scalar classes analysed — flat space, FRW, sourced, and self-referential sourcing — the $4\pi^2$ pairing is not generated by bulk scalar dynamics. Every route considered is phase-diagonal, consistent with the topological origin established in §4.

Section 6 presents a proof sketch connecting the Penrose/Bailey retarded cohomology class to the Hopf boundary generator ω_R via a residue argument. The key insight is that the canonical choice of γ as the celestial equator, combined with the causal support condition $\theta(x^0-y^0)$, forces the singularity of the twistor representative into the interior of the Hopf fiber contour. The resulting residue is $+1$ for the retarded sector and -1 for the advanced sector, completing the chain $\Psi_R, \Psi_A \to \omega_R, \omega_A \to 4\pi^2$ via Theorem 1. The argument is complete modulo one geometric lemma (§6.4) concerning the Hopf bundle topology, which is stated precisely and follows from standard results but requires explicit verification.

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Appendix A: Explicit Verification of the Phase Calculation

We verify that the spinor $\lambda_\alpha(\phi) = (1, e^{i\phi})^T$ returns to itself (not to its negative) after $\phi : 0 \to 2\pi$.

At $\phi = 0$: $\lambda_\alpha = (1,1)^T$.
At $\phi = 2\pi$: $\lambda_\alpha = (1, e^{2\pi i})^T = (1,1)^T$. ✓

No sign flip. This contrasts with the spatial rotation case: under a rotation of angle ψ about the z-axis, $\lambda_\alpha \to \mathrm{diag}(e^{-i\psi/2}, e^{i\psi/2})\lambda_\alpha$, giving $\lambda_\alpha \to -\lambda_\alpha$ at $\psi = 2\pi$.

The difference: the null-direction loop parametrises a path on S^2 (simply connected; $\pi_1(S^2) = 0$). The spatial rotation loop parametrises a path in $SO(3)$ (not simply connected; $\pi_1(SO(3)) = \mathbb{Z}/2$). The former does not exhibit spinor double-cover behaviour; the latter does.

Appendix B: Cohomology Computation via Mayer-Vietoris

We verify the Heegaard cohomology via the Mayer-Vietoris sequence for $S^3 = V_+ \cup_{T^2} V_-$.

The Mayer-Vietoris sequence in cohomology: $$\cdots \to H^k(S^3) \to H^k(V_+) \oplus H^k(V_-) \xrightarrow{i^*_+ - i^*_-} H^k(T^2) \to H^{k+1}(S^3) \to \cdots$$

***k* = 0:** $H^0(S^3) = \mathbb{Z}$, $H^0(V_\pm) = \mathbb{Z}$, $H^0(T^2) = \mathbb{Z}$. The sequence gives $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$, consistent.

***k* = 1:** $H^1(S^3) = 0$, $H^1(V_\pm) = \mathbb{Z}$, $H^1(T^2) = \mathbb{Z}^2$. The sequence gives: $$0 \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{i^*_+ - i^*_-} \mathbb{Z}^2 \to H^2(S^3) = 0$$

The map $i_+^* - i_-^* : \mathbb{Z}^2 \to \mathbb{Z}^2$ must be an isomorphism (since kernel = 0 from the left and cokernel = 0 from the right). This confirms that the images of i_+^* and i_-^* together generate all of $H^1(T^2) \cong \mathbb{Z}^2$, and they are complementary. ✓

***k* = 2:** $H^2(S^3) = 0$, $H^2(V_\pm) = 0$, $H^2(T^2) = \mathbb{Z}$. The sequence gives $0 \to \mathbb{Z} \to H^3(S^3) = \mathbb{Z} \to 0$, consistent. ✓

This confirms the setup of §4.2 via an independent method.

Appendix C: The Hopf Stereographic Correspondence

We record the explicit Hopf coordinate dictionary used in §6.4, for reference.

Lemma (Hopf Stereographic Map). *In the affine coordinate $\zeta = \lambda_1/\lambda_0$ on $\mathbb{CP}^1$, the Hopf* *projection $\pi : S^3 \to S^2$ is:* $$[\lambda_0:\lambda_1] \longmapsto \left(\frac{2\,\mathrm{Re}\,\zeta}{1+|\zeta|^2},\ \frac{2\,\mathrm{Im}\,\zeta}{1+|\zeta|^2},\ \frac{1-|\zeta|^2}{1+|\zeta|^2}\right) \in S^2.$$ *Hence: the equator corresponds to $|\zeta| = 1$; the northern hemisphere $n_3 > 0$ to $|\zeta| < 1$; the southern hemisphere $n_3 < 0$ to $|\zeta| > 1$.*

Proof. Normalise $(\lambda_0,\lambda_1) \in S^3 \subset \mathbb{C}^2$ as $(\lambda_0,\lambda_1) = (1+|\zeta|^2)^{-1/2}(1,\zeta)$ and substitute into the standard Hopf map $(\lambda_0,\lambda_1) \mapsto (2\,\mathrm{Re}(\lambda_0\bar{\lambda}_1), 2\,\mathrm{Im}(\lambda_0\bar{\lambda}_1), |\lambda_0|^2 - |\lambda_1|^2)$. Direct computation gives the formula above. The hemisphere correspondence follows immediately from $n_3 = (1-|\zeta|^2)/(1+|\zeta|^2)$. □

Corollary. The Hopf fiber $S_\lambda{}^1 = \{|\zeta| = 1\}$ over the canonical equatorial loop γ bounds the unit disk $|\zeta| < 1$, which corresponds to the future hemisphere V_+ of the sky sphere. Any spinor α with $[\alpha] \in V_+$ satisfies $|\alpha_1/\alpha_0| < 1$, so the pole of $\langle \lambda, \alpha \rangle$ at $\zeta = \alpha_1/\alpha_0$ lies strictly inside $S_\lambda{}^1$.

This is the explicit content of the Pole Location Lemma used in §6.4.

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