The Complexified Null Cone

Geometric Seat of Retrocausal Activation in the STF Framework

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Abstract

The Selective Transient Field (STF) is derived from first principles in [Paz 2026c] — from General Relativity (Peters formula), ghost-freedom constraints (DHOST Class Ia), cosmological boundary conditions, and 10D compactification, with no observational input. The resulting field equation admits both retarded and advanced solutions as a mathematical consequence of its wave structure; no additional assumption is required. The present paper establishes the geometric object that simultaneously encodes all advanced solutions: the conjugate regulus of the complexified null cone. Over $\mathbb{R}$, the null cone $k_\mu k^\mu = 0$ has signature (3,1) and contains no real projective lines. Over $\mathbb{C}$, it is doubly ruled, with the two ruling families parametrised by the two independent spinor factors λ_α and $\overset{\sim}{\lambda}_{\dot{\alpha}}$ in the decomposition $k_{\alpha \dot{\alpha}} = \lambda_{\alpha} \overset{\sim}{\lambda}_{\dot{\alpha}}$. We show by explicit spinor calculation that each factor accumulates a phase of 2π independently through one complete null rotation, their product is $4\pi^2$, and the $T^{2} = U (1)_{\lambda} \times U (1)_{\overset{\sim}{\lambda}}$ fiber of the complexified null cone is precisely the transaction closure torus whose fundamental group $\pi_1(T^2) = \mathbb{Z}^2$ sets the STF threshold [Paz 2026d]. The physical identification follows: Regulus 1 (λ_α family) encodes retarded solutions; the conjugate Regulus 2 ($\overset{\sim}{\lambda}_{\dot{\alpha}}$ family) encodes advanced solutions; every intersection point of the two reguli is a causal transaction. We compute the cross-ratio of four canonical reference points on $\mathbb{CP}^1$ and show that it equals $\sec^2(\alpha/2)$, degenerating to 1 at threshold (locked configuration in the sense of Shvalb-Medina [2026]) and separating two geometrically distinct regimes: above threshold, the four points lie on the complex unit circle (mobile, Bennett-type); below threshold, they collapse to a monotone real configuration (hypo-paradoxical, locked). The cross-ratio deviation $\sec^2(\alpha/2) - 1$ defines a geometric workspace identical in structure to the temporal workspace $\mathscr{W}_T$ of [Paz 2026e]. An explicit calculation shows that the Fubini-Study overlap of the tidal null directions does **not** recover the $(\tau/\tau^*)^{7/4}$ exponent of the workspace formula: the overlap follows a 5/8 (orbital phase) scaling rather than the 7/4 (curvature amplitude) scaling. We complete the picture by computing the full twistor-space ($\mathbb{CP}^3$) Fubini-Study transition probability and showing it also does not recover $(\tau/\tau^*)^{7/4}$: the near-threshold deviation is quadratic (ϵ^2) while the workspace formula is linear (ϵ), a structural obstruction independent of parameters. The projectively invariant combination $(|\mu_*|/|\mu(\tau)|)^7$ gives the exact restatement $\mathscr{W}_T = (|\mu_*|/|\mu(\tau)|)^7 - 1$ — a clean twistor-geometric translation of the workspace formula, not a derivation. The 7/4 exponent does not require a new geometric explanation: the decomposition $7 = 3 + 4$ — three from the Weyl tensor scaling $\mathscr{R} \sim a^{-3}$ and four from the Peters chain rule $\dot{a} \sim a^{-3}$ — shows that $\mathscr{D} = n^{\mu}\nabla_{\mu}\mathscr{R} \sim a^{-7}$ is already derived in [Paz 2026c] from DHOST ghost-freedom constraints and GR. The complexified null cone is the correct geometric seat of the threshold transition and the retrocausal structure; the amplitude scaling of the workspace is a derived consequence of the field equation, not a gap in the geometry.

Keywords: complexified null cone, conjugate regulus, spinor decomposition, retrocausality, STF threshold, cross-ratio, hypo-paradoxical linkages, causal transaction, Peters formula, temporal workspace

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1. Introduction

1.1 The Physical Situation

The STF field equation, derived from first principles in [Paz 2026c, Section III.A]:

$$ \Box\phi - m_{s}^{2} \phi = - \frac{\zeta}{\Lambda} \, n^{\mu} \nabla_{\mu} \mathcal{R} $$

is a massive wave equation with a curvature-rate source. Its derivation requires no observational input: four theoretical constraints — General Relativity (Peters formula), ghost-freedom (DHOST Class Ia), cosmological boundary conditions, and 10D compactification — uniquely determine the Lagrangian, the field mass $m_s = 3.94 \times 10^{-23}$ eV, and the coupling $\zeta/\Lambda \approx 1.35 \times 10^{11}$ m² [Paz 2026c].

Like all wave equations, the STF equation admits two families of Green's functions: retarded (G^{+}, causal) and advanced (G^{-}, retrocausal). In ordinary field theory, advanced solutions are discarded by the choice of initial conditions. The STF framework does not discard them. The reason is structural: the activation threshold $\mathscr{D}_{crit}$, derived in [Paz 2026c, §III.D] from causal coherence requirements in an expanding universe, requires the field to maintain bi-directional causal coupling — a condition that cannot be satisfied by the retarded solution alone. The $4\pi^2$ factor in $\mathscr{D}_{crit}$ encodes both a forward and a backward phase closure [Paz 2026c, §III.D.3]; the complexified null cone realises this as the product of two independent spinor windings — one holomorphic (Regulus 1, retarded) and one anti-holomorphic (Regulus 2, advanced). The advanced solution is not an optional add-on: the backward phase closure in the threshold condition requires negative-frequency modes, and those modes are precisely what the conjugate regulus encodes.

This paper asks: what is the natural geometric language for the set of all advanced solutions?

1.2 The Shvalb-Medina Connection

In the companion papers [Paz 2026d, 2026e], the STF threshold is mapped to the concept of a hypo-paradoxical linkage introduced by Shvalb and Medina [2026]. A hypo-paradoxical linkage is a closed kinematic chain that satisfies the Chebyshev-Grübler-Kutzbach mobility formula ($M > 0$) yet has a zero-dimensional configuration space — it is kinematically locked despite formally having degrees of freedom. The rigidity arises from a geometric alignment condition: all joint screw axes intersect a common line in monotone order, collapsing the configuration space. When this alignment is broken — when the ordering is non-monotone — the linkage becomes mobile (the Bennett case), and its joint axes lie on a doubly-ruled quadric surface, a hyperboloid of one sheet.

The STF structural parallel: below the activation threshold $\mathscr{D} < \mathscr{D}_{crit}$, the field is frozen — locked, hypo-paradoxical. Above threshold, it is active — mobile, Bennett-type. The doubly-ruled surface that appears in the Bennett linkage has a counterpart in the STF field: the complexified null cone.

1.3 The Main Claim

The conjugate regulus of the complexified null cone is the geometric object that simultaneously encodes all advanced solutions of the STF field equation. The two ruling families correspond to retarded and advanced propagation. The intersection of a line from each family at a spacetime event is a causal transaction — the geometric encoding of the handshake between future and past.

The threshold condition $\mathscr{D} = \mathscr{D}_{crit}$ is the condition under which this doubly-ruled structure becomes non-degenerate: below threshold, the structure collapses to the locked (real, monotone) configuration; at threshold, it is marginally degenerate; above threshold, it is genuinely complex and doubly-ruled, and transactions are physically realized.

Connection to the EXISTS/HAPPENS distinction. This geometric transition is the explicit physical realisation of the EXISTS/HAPPENS distinction established in [Paz 2026d, §1.2] and [General Theory V0.9]. EXISTS corresponds to the degenerate configuration: the complexified null cone has collapsed to a real monotone line, the four canonical reference points on $\mathbb{CP}^1$ are locked in sequential real order, the doubly-ruled structure is absent, and no transaction is possible. The causal trajectory is open — retarded propagation only, no closed loop, no topological interior. HAPPENS corresponds to the non-degenerate configuration: the four points lie on the complex unit circle, the complexified null cone is genuinely doubly-ruled, the Hopf torus is non-degenerate, the $(1, -1)$ Chern class pairing completes both 2π windings, the T^2 winding reaches $4\pi^2$, and the transaction closes. The causal loop is closed — it has a topological interior, whose measure is exactly the $4\pi^2$ fundamental domain of the T^2 fiber. The $4\pi^2$ is not only the threshold normalization: it is the measure of the interior that HAPPENS requires and EXISTS lacks. The complexified null cone is therefore the geometric grounding for that distinction — the paper that shows what it costs, geometrically, to cross from EXISTS to HAPPENS.

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2. The Complexified Null Cone Is Doubly Ruled

2.1 The Real Null Cone Is Not Ruled

The characteristic variety of the STF field equation — the surface in momentum space on which the principal symbol $\sigma(k) = k_\mu k^\mu$ vanishes — is the null cone:

$\mathscr{C} = \{k \in \mathbb{P}^3 : k_\mu k^\mu = 0\}$

Over $\mathbb{R}$, this is a quadric of signature (3,1) — three positive eigenvalues and one negative. A real projective quadric contains real projective lines if and only if it has signature (2,2). Since (3,1) $\neq$ (2,2), the real null cone **contains no real projective lines**. Over $\mathbb{R}$, there is no ruled structure, and the Shvalb-Medina framework — which requires a doubly-ruled quadric — cannot be directly applied.

2.2 The Complexified Null Cone Is Doubly Ruled

Over $\mathbb{C}$, every smooth quadric in $\mathbb{P}^3$ is doubly ruled. The complexified null cone:

$\mathscr{C}_{\mathbb{C}} = \{k \in \mathbb{CP}^3 : k_\mu k^\mu = 0\}$

admits two families of complex projective lines (the two reguli). These are parametrised by the spinor decomposition:

$$ k_{\alpha \dot{\alpha}} = k_{\mu} \left(\sigma^{\mu} \right)_{\alpha \dot{\alpha}} = \lambda_{\alpha} \overset{\sim}{\lambda}_{\dot{\alpha}} $$

where $\lambda_\alpha \in \mathbb{CP}^1$ and $\overset{\sim}{\lambda}_{\dot{\alpha}} \in \mathbb{C P}^{1}$ are independent complex spinors. **Regulus 1** is the family of lines obtained by holding $\overset{\sim}{\lambda}_{\dot{\alpha}}$ fixed and varying λ_α over $\mathbb{CP}^1$. **Regulus 2** is the family obtained by holding λ_α fixed and varying $\overset{\sim}{\lambda}_{\dot{\alpha}}$. Every line of Regulus 1 meets every line of Regulus 2 in exactly one point — the specific null direction $k_{\alpha \dot{\alpha}} = \lambda_{\alpha} \overset{\sim}{\lambda}_{\dot{\alpha}}$.

On the real slice ($\overset{\sim}{\lambda}_{\dot{\alpha}} = \overline{\lambda_{\alpha}}$), the decomposition reduces to the standard null vector; the two families coincide, and the ruled structure collapses.

2.3 Physical Identification of the Two Reguli

The STF field equation in the presence of a boundary condition at future time t^* (the merger) has two natural classes of solutions:

- **Retarded solutions:** sourced by $G^+(x,x')$, propagating from past to future. These correspond to standard causal responses of the field to the evolving orbital curvature. They are parametrised by the holomorphic spinor λ_α, which transforms under the standard (holomorphic) representation of $SL(2,\mathbb{C})$ and corresponds to positive-frequency modes [Penrose 1960].
- **Advanced solutions:** sourced by $G^-(x,x')$, propagating from the future boundary backward. These are the retrocausal solutions whose existence is required by the bi-directional closure structure of $\mathcal{D}_{crit}$ [Paz 2026c, §III.D.3]. They are parametrised by the anti-holomorphic spinor $\overset{\sim}{\lambda}_{\dot{\alpha}}$, which transforms under the conjugate representation of $SL(2,\mathbb{C})$ and corresponds to negative-frequency (advanced) modes [Penrose 1960].

Although the holomorphic/anti-holomorphic identification is standard in twistor theory for massless fields, the STF field is massive ($m_s = 3.94 \times 10^{-23}$ eV). The identification nevertheless holds at the characteristic variety: the mass term ${m_s}^2\phi$ is lower-order in the principal symbol and shifts the propagator pole to $k_\mu k^\mu = -{m_s}^2$, but does not alter the orientation of the null cone boundary. Outgoing null directions (retarded boundary conditions) and incoming null directions (advanced boundary conditions) are distinguished by the sign of k^0 relative to the future-pointing normal — a condition that is independent of m_s and holds on the null cone $k_\mu k^\mu = 0$ used here.

The intersection of the two reguli at a point $k_{\alpha \dot{\alpha}} = \lambda_{\alpha} \overset{\sim}{\lambda}_{\dot{\alpha}}$ is the geometric encoding of a **causal transaction**: a specific null event where retarded and advanced contributions jointly determine the field state. This is the Cramer transactional interpretation [Cramer 1986] in the language of classical projective geometry.

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3. The $4\pi^2$ Factor from the T^2 Fiber

3.1 Setup

The spinor decomposition is not unique: the rescaling $\lambda_\alpha \rightarrow e^{i\theta}\lambda_\alpha$, $\overset{\sim}{\lambda}_{\dot{\alpha}} \rightarrow e^{- i \theta} \overset{\sim}{\lambda}_{\dot{\alpha}}$

leaves $k_{\alpha\dot{\alpha}}$ unchanged. This is the $U(1)$ helicity freedom of the decomposition. The phase spaces of λ_α and $\overset{\sim}{\lambda}_{\dot{\alpha}}$ are therefore two independent circles:

$$ U (1)_{\lambda} \times U (1)_{\overset{\sim}{\lambda}} \cong T^{2} $$

This T^2 is the fiber of the complexified null cone over the space of real null directions S^2.

3.2 The Explicit Phase Calculation

Winding path. Parametrise a circle of null directions in the xy-plane by $\phi \in [0,2\pi]$:

$k^\mu(\phi) = (1, \cos\phi, \sin\phi, 0)$

The corresponding spinor matrix (Weyl representation, $k_{\alpha\dot{\alpha}} = k_\mu(\sigma^\mu)_{\alpha\dot{\alpha}}$):

$$ k_{\alpha \dot{\alpha}} (\phi) = \begin{pmatrix}1 & e^{- i \phi} \\ e^{+ i \phi} & 1\end{pmatrix} $$

Factored as $\lambda_{\alpha} \overset{\sim}{\lambda}_{\dot{\alpha}}$ with $\overset{\sim}{\lambda} = \bar{\lambda}$ (real slice):

$$ \lambda_{\alpha} (\phi) = \begin{pmatrix}1 \\ e^{i \phi}\end{pmatrix} , \quad \quad \overset{\sim}{\lambda}_{\dot{\alpha}} (\phi) = \begin{pmatrix}1 \\ e^{- i \phi}\end{pmatrix} $$

Phase of λ_α. As $\phi : 0 \to 2\pi$:

$$ \lambda_{\alpha} (0) = \begin{pmatrix}1 \\ 1\end{pmatrix} , \quad \quad \lambda_{\alpha} (2 \pi) = \begin{pmatrix}1 \\ e^{2 \pi i}\end{pmatrix} = \begin{pmatrix}1 \\ 1\end{pmatrix} $$

The second component $\lambda_2 = e^{i\phi}$ completes one full winding. The phase accumulated is:

$\Delta\phi_\lambda = 2\pi$

Phase of $\overset{\sim}{\lambda}_{\dot{\alpha}}$. As $\phi : 0 \to 2\pi$:

$$ \overset{\sim}{\lambda}_{\dot{\alpha}} (0) = \begin{pmatrix}1 \\ 1\end{pmatrix} , \quad \quad \overset{\sim}{\lambda}_{\dot{\alpha}} (2 \pi) = \begin{pmatrix}1 \\ e^{- 2 \pi i}\end{pmatrix} = \begin{pmatrix}1 \\ 1\end{pmatrix} $$

The second component $\overset{\sim}{\lambda}_{2} = e^{- i \phi}$ completes one full winding in the opposite direction. The phase accumulated is:

$$ \Delta \phi_{\overset{\sim}{\lambda}} = 2 \pi $$

Independence. The two windings live on orthogonal $U(1)$ factors of T^2: any rephasing $\lambda_\alpha \to e^{i\theta}\lambda_\alpha$, $\overset{\sim}{\lambda}_{\dot{\alpha}} \rightarrow e^{- i \theta} \overset{\sim}{\lambda}_{\dot{\alpha}}$ moves phase between the two factors without changing their individual winding numbers. The winding generator is $(1,-1) \in \pi_1(T^2) = \mathbb{Z}^2$ — once forward in the λ direction and once backward in the $\overset{\sim}{\lambda}$ direction.

Total phase:

$$ \boxed{\Delta \phi_{\text{total}} = \Delta \phi_{\lambda} \times \Delta \phi_{\overset{\sim}{\lambda}} = 2 \pi \times 2 \pi = 4 \pi^{2}} $$

3.3 Recovery of the STF Threshold Factor

The STF activation threshold derived in [Paz 2026c, §III.D.3] is:

$$ \mathcal{D}_{crit} = \frac{m_{s} M_{\mathrm{Pl}} H_{0}}{4 \pi^{2}} $$

where the $4\pi^2$ factor was derived from two independent phase-closure conditions: one temporal ($\Delta\phi_{\text{temporal}} = 2\pi$, one Compton oscillation) and one spatial ($\Delta\phi_{\text{spatial}} = 2\pi$, one causal loop across the Compton wavelength). The cascade paper [Paz 2026d] identified this as the topological content of $\mathcal{D}_{crit}$, attributing it to $\pi_1(T^2)$ of the transaction closure torus.

The spinor calculation above provides the explicit geometric realisation of this identification:

ABSTRACT (§III.D.3)	GEOMETRIC (THIS PAPER)
Temporal 2π closure	Phase of λ_α through one null rotation
Spatial 2π closure	Phase of $\overset{\sim}{\lambda}_{\dot{\alpha}}$ through one null rotation
Independence	Orthogonal $U(1)$ factors of T^2 fiber
$ 4 ^{2} = (2)^{2} $	Area of fundamental domain of T^2 fiber
$\pi_1(T^2)$ generator	Winding $(1,-1)$ of null rotation on T^2

The $4\pi^2$ is not a normalisation convention. It is the area of the fundamental domain of the T^2 fiber of the complexified null cone, traversed once per complete null rotation. The threshold condition $\mathcal{D} = \mathcal{D}_{crit}$ is the condition that this winding completes — that the transaction closes.

3.4 Canonical Status of the T^2 and the (1,−1) Winding

Two questions arise about the canonical status of the T^2 and its $(1,-1)$ winding. We address them precisely.

Is the T^2 forced by the null-direction loop geometry? Yes, canonically. The Hopf fibration $S^{1} \hookrightarrow S^{3} \xrightarrow{\pi} S^{2}$ provides the canonical map: for any closed loop $\gamma : S^1 \to S^2$ in the space of null directions, the preimage $\pi^{-1}(\gamma) \subset S^3 \simeq SU(2)$ is a torus $T^2 = S^1 \times S^1$. This follows from the triviality of principal S^1-bundles over S^1 ($H^2(S^1, \mathbb{Z}) = 0$): the restriction of the Hopf bundle to any circle in the base is necessarily trivial, so its total space over that circle is $S^1 \times S^1$. The T^2 is therefore canonically attached to the null-direction loop — forced once the loop is chosen, not introduced by hand. The specific loop of Section 3.2 (the equatorial circle of null directions in the xy-plane) is the equator of S^2, whose Hopf preimage is the Clifford torus in S^3.

Is the $(1,-1)$ winding canonical? Yes, at the level of bundle topology — which is the appropriate level for this structure. One might ask whether the $(1,-1)$ is a holonomy of the canonical Hopf connection around the equatorial loop. It is not: the canonical Hopf connection (induced by the round metric on S^3) has holonomy $e^{i\pi} = -1$ around the equator — a phase of π, not 2π. But holonomy is not the relevant structure here, and the π vs 2π mismatch is not an obstacle.

The relevant structure is the transition-function winding number — the first Chern class of the bundle — which is connection-independent. The standard Hopf bundle has transition function $e^{i\phi}$ with winding degree $+1$ across the equatorial overlap; the conjugate (anti-Hopf) bundle has transition function $e^{-i\phi}$ with winding degree -1. The pair $\left(\lambda_{\alpha} , \overset{\sim}{\lambda}_{\dot{\alpha}} \right)$ lives in the Hopf and anti-Hopf bundles respectively: λ_α in the holomorphic sector (Regulus 1, retarded), $\overset{\sim}{\lambda}_{\dot{\alpha}}$ in the anti-holomorphic sector (Regulus 2, advanced). The $(1,-1)$ generator of $\pi_1(T^2) = \mathbb{Z}^2$ is canonically realised as the pair of transition-function winding degrees of these two bundles. This identification requires no choice of connection.

This is also the natural structure for the propagator bridge: in the twistor literature, the retarded/advanced distinction for sourced fields is implemented by contour selection and analytic continuation in twistor space — a relative cohomology phenomenon — not by parallel transport on an auxiliary bundle [Penrose TN14; Bailey TN14]. The canonical T^2 winding structure is precisely the topological scaffolding that contour integrals can see through winding numbers and residues (via $2\pi i$ factors).

The $2\pi^2$ vs $4\pi^2$ relationship. The Clifford torus embedded in unit S^3 has induced geometric area $2\pi^2$, while the flat coordinate fundamental domain (parametrised by $(\theta,\varphi) \in [0, 2\pi) \times [0, 2\pi))$ has area $4\pi^2 = (2\pi)^2$. The factor of two is metric normalisation: embedding the Clifford torus in unit S^3 scales each circle radius by $1 / \sqrt{2}$, halving the geometric area. The $4\pi^2$ in the threshold formula is the flat coordinate area; $2\pi^2 = \mathrm{Vol}(SU(2)) = \mathrm{Vol}(S^3)$ is the induced geometric area. These are different normalisations of the same topological object, not evidence of a double cover.

4. The Cross-Ratio and the Locked/Mobile Transition

4.1 Four Reference Points on $\mathbb{CP}^1$

The winding path $\lambda(\phi) = (1, e^{i\phi})^T$ on $\mathbb{CP}^1$ has affine coordinate $z(\phi) = e^{i\phi}$. Four reference points at $\phi = 0, \alpha, \pi, \alpha + \pi$ give:

$$z_1 = 1, \quad z_2 = e^{i\alpha}, \quad z_3 = -1, \quad z_4 = -e^{i\alpha}$$

Two antipodal pairs: (z_1, z_3) and (z_2, z_4).

4.2 The Cross-Ratio — Exact Computation

$$\text{cr} \left(z_{1} , z_{2} ; z_{3} , z_{4} \right) = \frac{\left(z_{1} - z_{3} \right) \left(z_{2} - z_{4} \right)}{\left(z_{2} - z_{3} \right) \left(z_{1} - z_{4} \right)}$$

Numerator: $(1-(-1))(e^{i\alpha}-(-e^{i\alpha})) = 2 \cdot 2e^{i\alpha} = 4e^{i\alpha}$.

Denominator: $(e^{i\alpha}-(-1))(1-(-e^{i\alpha})) = (1+e^{i\alpha})^2$.

$$\text{cr} (\alpha) = \frac{4 e^{i \alpha}}{\left(1 + e^{i \alpha} \right)^{2}} = \frac{4}{e^{- i \alpha} + 2 + e^{i \alpha}} = \frac{4}{2 + 2 \cos \alpha}$$

Applying $1 + \cos\alpha = 2\cos^2(\alpha/2)$:

$$\boxed{\text{cr} (\alpha) = \sec^{2} \left(\frac{\alpha}{2} \right)}$$

This is real and exact. The cross-ratio of these four symmetrically placed points is a pure real trigonometric function of the half-angle. All imaginary parts cancel.

4.3 The Degenerate Configuration at Threshold

At $\alpha = 0$: $z_1 = z_2 = 1$, $z_3 = z_4 = -1$. Both pairs collapse to single points. The cross-ratio:

$$\text{cr}(0) = \sec^2(0) = 1$$

This is exactly the degenerate case: when distinct points coincide, the cross-ratio loses its projective content. This is the $\mathbb{CP}^1$ realisation of the Shvalb-Medina aligned configuration — all joints collapsing to a common line in monotone order, configuration space dimension zero, hypo-paradoxical (locked).

The deviation from threshold:

$$ \text{cr} (\alpha) - 1 = \tan^{2} \left(\frac{\alpha}{2} \right) $$

This is 0 at $\alpha = 0$ and positive for $\alpha \neq 0$ — it opens continuously from zero, just as the workspace $\mathscr{W}_T$ opens from zero at $\tau = \tau^*$.

4.4 The Three Regimes — Complete Classification

The cross-ratio $\sec^2(\alpha/2)$ classifies the configuration of the four points for all values of α:

Above threshold ($\tau < \tau^*$, $\mathscr{D} > \mathscr{D}_{crit}$): real $\alpha > 0$.

The four points $\{1, e^{i\alpha}, -1, -e^{i\alpha}\}$ lie on the complex unit circle. They are distinct, genuinely complex, non-collinear. The cross-ratio exceeds 1. This is the mobile (Bennett-type) configuration: the joint axes of the Shvalb-Medina regulus are genuinely complex — not aligned. The conjugate regulus is non-degenerate, and physical transactions (advanced-retarded intersections) exist.

At threshold ($\tau = \tau^*$, $\mathscr{D} = \mathscr{D}_{crit}$): $\alpha = 0$, cross-ratio $= 1$.

Both pairs collapse. Geometric degeneracy. This is the boundary between the two regimes — the hypo-paradoxical point where $\mathscr{W}_T = 0$ and the transaction is marginally possible.

Below threshold ($\tau > \tau^*$, $\mathscr{D} < \mathscr{D}_{crit}$): imaginary $\alpha = i\beta$, $\beta > 0$.

The four points become $\{1, e^{-\beta}, -1, -e^{-\beta}\} \subset \mathbb{R}$. All four are real, and they appear in the monotone order $-1 < -e^{-\beta} < e^{-\beta} < 1$ along the real line. The cross-ratio:

$\text{cr}(i\beta) = \text{sech}^2(\beta/2) \in (0,1)$

This is the Shvalb-Medina locked configuration: all points on a common real line in sequential monotone order. This is precisely Proposition 1 of [Shvalb-Medina 2026] — a linkage whose screws all intersect a common line in monotone order is hypo-paradoxical. Below threshold, the four null reference points are in exactly this configuration. The conjugate regulus has degenerated to a real line — no complex doubly-ruled structure, no transactions.

REGIME	\$ \$	FOUR POINTS	CONFIGURATION	\$ ( ) \$
$\tau < \tau^*$ (active)	real, > 0	complex unit circle	mobile (Bennett)	\$ > 1 \$
$\tau = \tau^*$ (threshold)	\$ = 0 \$	pairs collapse	hypo-paradoxical	\$ = 1 \$
$\tau > \tau^*$ (inactive)	imaginary $i\beta$	monotone real line	locked (aligned)	\$ (0 , 1) \$

The threshold is the geometric event at which the winding path transitions from a complex circle to a real line. This transition is the projective-geometric realisation of the STF activation.

4.5 Geometric Workspace

Define the geometric workspace as:

$$ \mathcal{W}_{\text{geom}} (\alpha) \equiv \text{cr} (\alpha) - 1 = \tan^{2} \left(\frac{\alpha}{2} \right) $$

By construction, $\mathcal{W}_{\text{geom}}$ is 0 at threshold, positive above, and — for imaginary $\alpha = i\beta$ — becomes $-\tanh^2(\beta/2) \in (-1,0)$ below threshold. The structural form matches the temporal workspace of [Paz 2026e]:

$$ \mathcal{W}_{T} (\tau) = \left(\frac{\tau^{*}}{\tau} \right)^{7 / 4} - 1 $$

In both cases the workspace is zero at threshold, positive above, and bounded below by -1. The question of whether $\mathcal{W}_{\text{geom}}$ and $\mathcal{W}_T$ are numerically identical requires an identification between the winding angle α and the Peters curvature-rate ratio. We address this directly in the following section, with an explicit calculation.

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5. The 5/8 vs 7/4 Diagnostic

5.1 Setup — What Would the Identification Require

The identification $\mathcal{W}_{\text{geom}}(\alpha) = \mathcal{W}_T(\tau)$ requires:

$$ \sec^{2} \left(\frac{\alpha}{2} \right) = \left(\frac{\tau^{*}}{\tau} \right)^{7 / 4} $$

equivalently:

$$ \cos^{2} \left(\frac{\alpha}{2} \right) = \left(\frac{\tau}{\tau^{*}} \right)^{7 / 4} $$

The left side is the Fubini-Study transition probability between the threshold spinor $\lambda_0 = (1,1)^T$ and the wound spinor $\lambda(\alpha) = (1,e^{i\alpha})^T$:

$$ P (\alpha) = \frac{\left| \langle \lambda_{0} \right| \lambda (\alpha) \rangle |^{2}}{\left| \lambda_{0} \right|^{2} \left| \lambda (\alpha) \right|^{2}} = \frac{\left| 1 + e^{i \alpha} \right|^{2}}{4} = \cos^{2} \left(\frac{\alpha}{2} \right) $$

So the question becomes: is the Fubini-Study overlap between the threshold null direction and the evolved null direction equal to $(\tau/\tau^{*})^{7/4}$?

5.2 Identifying the Physical Null Direction Along the Inspiral

The clock vector n^{μ} is constant in direction at leading post-Newtonian order (it is approximately (1,0,0,0) in the center-of-mass frame, with corrections of order $v/c \lesssim 0.026$ at threshold). Its spinor decomposition is constant; it yields a trivial overlap of 1 at all τ.

The physically evolving spinor is carried by the tidal field. The Weyl tensor of the binary at orbital phase ϕ_{orb} has principal null direction (PND) in the orbital plane:

$$ k^{\mu_{+}} \left(\phi_{\text{orb}} \right) = \frac{1}{\sqrt{2}} \left(1 , \ \cos \phi_{\text{orb}} , \ \sin \phi_{\text{orb}} , \ 0 \right) \Longrightarrow \lambda_{\alpha} \left(\phi_{\text{orb}} \right) = \begin{pmatrix}1 \\ e^{i \phi_{\text{orb}}}\end{pmatrix} $$

Setting the reference at threshold ($\phi_{\text{orb}}(\tau^{*}) = 0$), the Fubini-Study overlap is:

$$ P (\tau) = \cos^{2} \left(\frac{\phi_{\text{orb}} (\tau)}{2} \right) $$

5.3 Orbital Phase from Peters

From Peters: $a(\tau) = a^{*}(\tau/\tau^{*})^{1/4}$, giving orbital frequency $\Omega(\tau) = \Omega^{*}(\tau^{*}/\tau)^{3/8}$. The orbital phase accumulated from threshold to time $\tau < \tau^{*}$:

$$ \phi_{\text{orb}} (\tau) = \int_{\tau}^{\tau^{*}} \Omega (\tau ') \, d \tau ' = \frac{8 \, \Omega^{*} \tau^{*}}{5} \left\lbrack 1 - \left(\frac{\tau}{\tau^{*}} \right)^{5 / 8} \right\rbrack $$

Therefore:

$$ P (\tau) = \cos^{2} \left(\frac{4 \, \Omega^{*} \tau^{*}}{5} \left\lbrack 1 - \left(\frac{\tau}{\tau^{*}} \right)^{5 / 8} \right\rbrack \right) $$

5.4 The Diagnostic Result — 5/8 Is Not 7/4

At threshold ($\tau = \tau^{*} - \epsilon$, small ϵ):

$$ P (\tau) \approx 1 - \frac{\left(\Omega^{*} \epsilon \right)^{2}}{4} = 1 - O \left(\epsilon^{2} \right) $$

$$ \left(\frac{\tau}{\tau^{*}} \right)^{7 / 4} \approx 1 - \frac{7 \epsilon}{4 \tau^{*}} = 1 - O (\epsilon) $$

The PND overlap decays as $O(\epsilon^{2})$ near threshold; the workspace ratio decays as $O(\epsilon)$. They have different leading-order behaviour.

At merger ($\tau \to 0$):

$$ P (\tau) \rightarrow \cos^{2} \left(\frac{4 \, \Omega^{*} \tau^{*}}{5} \right) = \text{constant} $$

$$ \left(\frac{\tau}{\tau^{*}} \right)^{7 / 4} \rightarrow 0 $$

The PND overlap saturates at a constant (the cosine of the total accumulated orbital phase); the workspace ratio goes to zero. They have structurally incompatible far-field behaviour.

The exponent mismatch. The 5/8 enters as:

$\phi_{\text{orb}} \propto \int \Omega \, d\tau \propto \int \tau^{-3/8} \, d\tau \propto \tau^{5/8}$

The 7/4 enters as:

$\mathscr{D}(\tau) \propto a^{-7} \propto (\tau^{1/4})^{-7} = \tau^{-7/4}$

These are two different functionals of the Peters inspiral:

EXPONENT	ORIGIN	PHYSICAL MEANING	GEOMETRIC OBJECT
$ 5 / 8 $	$, d $	Angular position of orbital phase	Fubini-Study distance on $\mathbb{CP}^1$
$ 7 / 4 $	a^{-7} tidal rate	Amplitude of curvature rate $\mathscr{D}$	Amplitude ratio; requires different object

5.5 Interpretation

The 5/8 exponent is captured by the complexified null cone geometry because the Fubini-Study metric on $\mathbb{CP}^1$ measures **angular separation** between directions. The Peters orbital phase is precisely the angular position of the tidal PND in the orbital plane; it naturally lives on $\mathbb{CP}^1$.

The 7/4 exponent is an **amplitude ratio** — the ratio $\mathscr{D}(\tau)/\mathscr{D}(\tau^*)$ of curvature rates. The Fubini-Study metric is projectively invariant: it is blind to the overall scale of λ_α and $\overset{\sim}{\lambda}_{\dot{\alpha}}$, and therefore blind to amplitude information. Amplitude ratios require a different geometric object — one that is not projectively invariant.

This is a structural result: the complexified null cone, as a projective variety, captures the directional (angular) content of the Peters dynamics. The amplitude content, which sets the 7/4 exponent and the temporal workspace $\mathscr{W}_T$, lies outside the projective structure. Section 5.6 below shows where it does live.

The distinction is not a failure of the geometric picture — the locked/mobile transition, the $4\pi^2$ identification, the Shvalb-Medina correspondence, and the cross-ratio classification all hold unconditionally and are independent of which exponent governs the workspace. The complexified null cone is the correct geometric seat of the retrocausal structure; the workspace amplitude scaling is a derived consequence of the field equation.

5.6 The Full Twistor Space and the Exact Workspace Restatement

Full twistor space $\mathbb{PT} = \mathbb{CP}^3$ encodes both a null direction $\lambda^\alpha \in \mathbb{CP}^1$ and a displacement $\mu^{\dot{\alpha}} = ix^{\alpha\dot{\alpha}}\lambda_\alpha$, where $x^{\alpha\dot{\alpha}}$ is the spacetime position. Along the Peters inspiral, $|\mu^{\dot{\alpha}}| \sim a(\tau)|\lambda|$, so μ carries the orbital separation. The $\mathbb{CP}^3$ Fubini-Study transition probability between the threshold twistor Z_* and the inspiral twistor $Z(\tau)$ is:

$$ P_{\mathrm{FS}} (\tau) = \frac{\left| Z_{*} \cdot \bar{Z} (\tau) \right|^{2}}{\left| Z_{*} \right|^{2} \, \left| Z (\tau) \right|^{2}} $$

Explicit computation. With $\lambda_\alpha = (1,1)^T$ held fixed, $\mu^{\dot{\alpha}}(\tau) = ia(\tau)(1,1)^T$, and $u \equiv (\tau/\tau^*)^{1/4}$ so that $a(\tau) = a_* u$:

$$ Z_{*} \cdot \bar{Z} (\tau) = \underbrace{2}_{\lambda \text{ overlap}} + \underbrace{2 a_{*}^{2} u}_{\mu \text{ cross-term}} $$

$$ P_{\mathrm{FS}} (u) = \frac{\left(1 + a_{*}^{2} u \right)^{2}}{\left(1 + a_{*}^{2} \right) \left(1 + a_{*}^{2} u^{2} \right)} $$

The deviation from threshold ($u = 1$):

$$ 1 - P_{\mathrm{FS}} (u) = \frac{a_{*}^{2} (u - 1)^{2}}{\left(1 + a_{*}^{2} \right) \left(1 + a_{*}^{2} u^{2} \right)} $$

Near-threshold behaviour ($u = 1 - \epsilon$, $\epsilon \ll 1$):

$$ 1 - P_{\mathrm{FS}} \approx \frac{\epsilon^{2}}{a_{*}^{2}} = O \left(\epsilon^{2} \right) $$

$$ 1 - \left(\frac{\tau}{\tau^{*}} \right)^{7 / 4} \approx \frac{7 \epsilon}{4} = O (\epsilon) $$

The deviation of P_{FS} from threshold is **quadratic**; the workspace formula is **linear**. The coefficient of ϵ^1 in $1 - P_{\mathrm{FS}}$ is exactly zero — P_{FS} has a maximum at threshold with vanishing first derivative. This is a structural obstruction: no choice of parameters, coordinates, or limit resolves the mismatch.

Midpoint check ($u = 1/2$, $a_* = 730$): $1 - P_{\mathrm{FS}} \approx 1.9 \times 10^{-6}$; $1 - u^7 \approx 0.992$. Approximately five to six orders of magnitude apart.

The exact twistor restatement. Although P_{FS} does not recover the workspace formula, the projectively invariant ratio $(|\lambda|/|\mu|)^7$ does — exactly, not asymptotically. Since $|\mu(\tau)| = a(\tau)|\lambda|$:

$$ \frac{| \lambda |^{7}}{\left| \mu (\tau) \right|^{7}} = a (\tau)^{- 7} , \quad \quad \frac{\left(| \lambda | / \left| \mu (\tau) \right| \right)^{7}}{\left(| \lambda | / \left| \mu_{*} \right| \right)^{7}} = \left(\frac{a_{*}}{a (\tau)} \right)^{7} = \left(\frac{\tau^{*}}{\tau} \right)^{7 / 4} $$

Therefore:

$$ \boxed{\mathcal{W}_{T} (\tau) = \left(\frac{\left| \mu_{*} \right|}{\left| \mu (\tau) \right|} \right)^{7} - 1} $$

This is an exact identity: **the temporal workspace is the seventh power of the inverse ratio of μ-norms at threshold and at time τ**. The workspace formula [Paz 2026e] is a restatement, in twistor language, of the ratio of orbital separations raised to the seventh power. This is a translation, not a derivation — the exponent 7 is not produced by the twistor geometry but is an input from the curvature-rate scaling.

5.7 Why $n = 7$ Is Already Derived in V7.0

The exponent $n = 7$ in $\mathscr{D} \sim a^{-7}$ does not require a new geometric explanation. It decomposes as $7 = 3 + 4$:

- **3:** The Weyl curvature of a binary scales as $\mathscr{R} \sim GM/a^3$ — the tidal field of two point masses at separation a. This is standard GR.
- **4:** The Peters radiation-reaction rate $\dot{a} \sim -Ca^{-3}$ (at 2.5 post-Newtonian order) contributes a factor of a^{-4} via the chain rule $d\mathscr{R}/dt = (\partial\mathscr{R}/\partial a)\dot{a} \sim (-3a^{-4})(-Ca^{-3}) \sim a^{-7}$.

Combined: $\mathscr{D} = n^{\mu}\nabla_{\mu}\mathscr{R} \sim a^{-7}$.

The coupling structure $n^{\mu}\nabla_{\mu}\mathscr{R}$ is fixed by DHOST ghost-freedom constraints [Paz 2026c, Section III]: it is the unique lowest-order operator coupling a scalar to the *rate* of curvature change. Given this coupling, GR determines $\mathscr{R} \sim a^{-3}$ and Peters determines $\dot{a} \sim a^{-3}$; the chain rule gives $n = 7$ with no free parameters. The exponent is in V7.0, not in the projective geometry of the null cone.

The geometric picture of Thread 3 is therefore complete and self-contained: the complexified null cone encodes the retrocausal structure and the threshold transition; the workspace amplitude scaling is a downstream consequence of the field equation source term, already derived from first principles.

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6. The Complete Picture

The following diagram summarises the established correspondences.

STF Threshold as Projective Geometric Transition

PHYSICS	PROJECTIVE GEOMETRY ($\mathbb{CP}^1$)	KINEMATICS (SHVALB-MEDINA)
$\mathcal{D} < \mathcal{D}_{crit}$ (inactive)	$\alpha = i\beta$; four points on real line, monotone	Hypo-paradoxical, locked
$\mathcal{D} = \mathcal{D}_{crit}$ (threshold)	$\alpha = 0$; pairs collapse, cr = 1	Marginally hypo-paradoxical
$\mathcal{D} > \mathcal{D}_{crit}$ (active)	α real, > 0; four points on complex circle	Mobile, Bennett-type

Retarded/Advanced Identification

RULING FAMILY	SPINOR	SOLUTION TYPE	PHYSICS
Regulus 1	λ_α (holomorphic)	Retarded G^+	Causal propagation
Regulus 2	$\overset{\sim}{\lambda}_{\dot{\alpha}}$ (anti-holomorphic)	Advanced G^-	Retrocausal (theoretically required by $\mathcal{D}_{crit}$ structure)
Intersection	$ k_{} = {} {} $	Transaction	Handshake event

Geometric Layering — What Each Level Captures

GEOMETRIC OBJECT	PETERS CONTENT	EXPONENT	STATUS
$\mathbb{CP}^1$ null direction (Thread 3)	Orbital phase (angular)	$ 5 / 8 $	Established ✓
$\mathbb{CP}^3$ P_{FS} (Thread 4)	Displacement + direction; cancels	structural obstruction	Not 7/4 ✗
$(\lvert\mu_*\rvert/\lvert\mu(\tau)\rvert)^7$ (Thread 5)	Amplitude ratio (exact restatement)	7/4 (exact)	V7.0 ✓
Field equation source $n^\mu \nabla_\mu \mathcal{R}$	Curvature amplitude (7 = 3 + 4)	7/4 (derived)	V7.0 ✓

The $4\pi^2$ Chain

$$ 4 \pi^{2} = \underset{\underset{\Delta \phi_{\lambda}}{\underbrace{}}}{2 \pi} \times \underset{\underset{\Delta \phi_{\overset{\sim}{\lambda}}}{\underbrace{}}}{2 \pi} = \text{Area of } T^{2} \text{ fiber} = \pi_{1} \left(T^{2} \right) \text{ generator traversed once} $$

This closes the chain: $4\pi^2$ in the threshold formula [Paz 2026c, D.3] → $\pi_1(T^2)$ identification [Paz 2026d] → T^2 fiber of complexified null cone → product of independent spinor windings (this paper).

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7. Open Problems

The Fubini-Study 5/8 **exponent as an observable.** The orbital phase scaling $\phi_{\text{orb}} \propto \tau^{5/8}$ is a prediction of the complexified null cone picture: the Fubini-Study distance between the threshold PND and the evolved PND grows as $\tau^{5/8}$ above threshold. This is a calculable geometric observable — in principle distinguishable from the 7/4 amplitude scaling in systems where orbital phase and curvature amplitude can be measured independently. The two exponents characterise different measurable aspects of the same inspiral: the orientation of the tidal field (5/8) and the magnitude of the curvature rate (7/4).

The transaction density. The density of intersection points between lines of the two reguli as a function of the winding angle α is not computed here. This density — the number of transactions per unit spacetime volume as a function of τ/τ^* — is a natural prediction of the framework and would connect Thread 3 to the observational UHECR statistics.

The propagator bridge: established via Hopf-coordinate residue argument. The problem of connecting the geometric transaction closure to the propagator product $G^+(x,x') \cdot G^-(x',x)$ has been resolved via an explicit residue calculation. The bridge is established conditional on the retarded Green function's twistor representative having a simple pole at $\langle\lambda, \alpha\rangle = 0$ — a property that is standard in the Penrose-Bailey relative cohomology framework [Penrose TN14; Bailey TN14] and is the basis of the argument below.

What has been established. An exhaustive calculation through all scalar field routes confirms that the $4\pi^2$ factor cannot be generated dynamically: flat space, FRW expansion, the STF source term $n^\mu\nabla_\mu\mathscr{R}$, the self-referential \$n^{\mu} = \nabla^{\mu} \phi / \sqrt{2 X}\$, and the merger boundary condition at t^* — all are phase-diagonal. The Friedmann energy condition $\rho_\phi = \rho_{crit}$ gives the correct scaling $m_s M_{\text{Pl}} H$ but requires $\kappa \sim 97$; the 10D-derived coupling is $\kappa \sim 10^{70}$ — a mismatch of 68 orders of magnitude. The $4\pi^2$ is topological, confirmed from both sides.

The topological transgression theorem. The Heegaard splitting $S^3 = V_+ \cup_{T^2} V_-$ with $V_\pm \cong S^1 \times D^2$ gives the proved chain:

\$\$ H^{1} \left(V_{+} \right) \oplus H^{1} \left(V_{-} \right) \xrightarrow{\text{restriction}} H^{1} \left(T^{2} \right) \cong \mathbb{Z}^{2} \xrightarrow{\cup} H^{2} \left(T^{2} \right) \xrightarrow{\int} 4 \pi^{2} \$\$

The residue mechanism. The apparent obstruction — that homogeneity -2 and winding $+1$ are not automatically identified — is resolved by the causal support condition. The key steps are:

1. **Canonical γ:** Define the loop γ as the celestial equator $\{k^\mu : g(k,\partial_t) = 0\}$, which canonically divides the sky sphere into future hemisphere V_+ and past hemisphere V_-.

2. **Causal forcing:** For any retarded pair (x,y) with $y \prec x$, the null direction α from y to x satisfies $[\alpha] \in V_+$ by definition of future-directed.

3. **Pole Location Lemma (proved):** In the Hopf affine coordinate $\zeta = \lambda_1/\lambda_0$, the Hopf map gives $n_3 = (1-|\zeta|^2)/(1+|\zeta|^2)$, so $[\alpha] \in V_+$ iff $|\alpha_1/\alpha_0| < 1$. The pole of $\langle\lambda, \alpha\rangle = \lambda_0(\alpha_1 - \zeta\alpha_0)$ is at $\zeta = \alpha_1/\alpha_0$, which lies strictly inside the unit circle $S^1{}_\lambda = \{|\zeta| = 1\}$.

4. **Residue:** Since the pole is inside the contour: $\frac{1}{2 \pi i} \oint_{S^{1_{\lambda}}} \frac{- \alpha_{0} \, d \zeta}{\alpha_{1} - \zeta \alpha_{0}} = + 1 \quad \Longrightarrow \quad \left\lbrack \Psi_{R} \right\rbrack |_{T}^{2_{\gamma}} = \left\lbrack i \, d \theta \right\rbrack = \left\lbrack \omega_{R} \right\rbrack$

The mirror argument for the advanced sector gives $[\Psi_A]|_{T^2{}_\gamma} = [\omega_A]$. Combined with the transgression theorem:

$\int_{T^2{}_\gamma} \Psi_R \wedge \Psi_A = \int_{T^2{}_\gamma} \omega_R \wedge \omega_A = 4\pi^2$

The complete chain is: $y \prec x \Rightarrow [\alpha] \in V_+ \Rightarrow |\alpha_1/\alpha_0| < 1 \Rightarrow$ pole inside $S^1{}_\lambda \Rightarrow$ residue $= +1 \Rightarrow [\Psi_R]|_{T^2{}_\gamma} = [\omega_R] \Rightarrow 4\pi^2$

The key insight: the homogeneity -2 / winding $+1$ mismatch is not an obstruction — it is resolved by the causal support condition, which is precisely what places the pole in a definite hemisphere and gives the residue its sign.

Significance for the STF framework. This result strengthens the framework's internal consistency in three specific ways. First, the threshold becomes self-consistent in a stronger sense: previously, the $4\pi^2$ threshold normalization was derived from a topological closure condition and separately the field equation admitted retarded and advanced solutions — now the topological closure condition that defines the threshold is the same $4\pi^2$ that the field equation's own propagator structure satisfies, from two sides. Second, the EXISTS/HAPPENS distinction acquires analytic grounding: EXISTS = field propagates retarded-only = no closed causal loop = no $4\pi^2$ completion; HAPPENS = retarded and advanced Green functions pair on the Hopf torus = residues $\pm 1 = 4\pi^2$ = threshold crossed. Third, the two main pillars of the framework — the threshold derivation and the null-cone geometry — were previously connected by a well-motivated but unproved assertion; they are now connected by an explicit chain of proved steps. In practical terms, STF changes from a framework with a striking topological idea and a missing bridge, to a framework with a proved topological backbone and a much narrower remaining analytic dependence. The null-cone sector now constitutes a standalone mathematical-physics core whose most distinctive normalization, $4\pi^2$, is a genuine topological consequence of the null-cone geometry rather than a loose interpretive insertion.

The two-sector architecture has direct precedent: Hughston and Hurd [HH TN] show that for any positive-frequency massive state of mass m and spin s, there exists a unique two-point field massless in each variable separately — the “massive object from two null sectors” structure the STF two-regulus construction realises geometrically. The $n = 7$ exponent is already derived in V7.0 from the DHOST coupling and GR chain rule.

Status. The propagator bridge is established conditional on the Penrose-Bailey singularity structure. The Pole Location Lemma and residue calculation are proved explicitly. This argument is developed in full in the companion standalone paper [Paz 2026, *Topological Closure on the Complexified Null Cone*, V5.0] and posted as a public answer to MathOverflow question 509131.

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8. Conclusion

The STF field’s advanced solutions are geometrically encoded by the conjugate regulus of the complexified null cone. The physical identification — Regulus 1 (retarded), Regulus 2 (advanced), intersection (transaction) — follows from the spinor decomposition $k_{\alpha \dot{\alpha}} = \lambda_{\alpha} \overset{\sim}{\lambda}_{\dot{\alpha}}$ of null directions. The $4\pi^2$ factor in the STF activation threshold is the area of the T^2 fiber of this complexified null cone, traversed once per complete null rotation by the product of two independent 2π spinor windings. The threshold crossing is geometrically identical to the Shvalb-Medina transition from aligned (locked) to non-aligned (mobile) regulus: below threshold, the four canonical reference points on $\mathbb{CP}^1$ lie on a real monotone line; above threshold, they lie on a complex circle. The cross-ratio $\sec^2(\alpha/2)$ is the projective invariant encoding this transition.

The 5/8 vs 7/4 exponent diagnostic — extended through the full $\mathbb{CP}^3$ twistor calculation — establishes that projective geometry captures the directional aspect of Peters dynamics (5/8, orbital phase) but not the amplitude aspect (7/4, curvature rate). This is not a gap: the workspace formula $\mathscr{W}_T = (|\mu_*|/|\mu(\tau)|)^7 - 1$ is an exact twistor restatement of the Peters amplitude ratio, and the exponent $7 = 3 + 4$ is already derived in V7.0 from the DHOST coupling structure and GR chain rule. The complexified null cone provides the geometric seat of the retrocausal activation; the workspace amplitude is a consequence of the field equation source term. The picture is complete. The complexified null cone is also the explicit geometric grounding for the EXISTS/HAPPENS distinction [Paz 2026d, §1.2]: EXISTS is the degenerate real configuration — locked, no doubly-ruled structure, no transaction, no interior; HAPPENS is the non-degenerate complex configuration — mobile, doubly-ruled, transaction completing, T^2 winding reaching $4\pi^2$, interior formed. The $4\pi^2$ threshold normalization and the $4\pi^2$ interior measure are the same quantity seen from two sides: the cost of the transition and the content of what the transition produces.

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Appendix A: Why the Spinor Double-Cover Does Not Affect the $4\pi^2$ Result

A Dirac spinor acquires a sign flip $\psi \to -\psi$ under a 2π spatial rotation and only returns to itself under 4π (the spinor double cover $SU(2) \to SO(3)$). This is a separate phenomenon from the winding calculated in Section 3.

The distinction is in the path. Spatial rotation by angle ψ about the z-axis acts on λ_α as:

$$ \lambda_{\alpha} \rightarrow \begin{pmatrix}e^{- i \psi / 2} & 0 \\ 0 & e^{+ i \psi / 2}\end{pmatrix} \lambda_{\alpha} $$

Under $\psi : 0 \to 2\pi$, this gives $\lambda_\alpha \to -\lambda_\alpha$ — the double-cover sign flip.

The path in Section 3 is different: we rotate the **null direction** k^μ in the xy-plane, not the spatial frame. The spinor $\lambda_\alpha(\phi) = (1, e^{i\phi})^T$ returns to itself after $\phi : 0 \to 2\pi$ — no sign flip. This is because the path traces a generator of $\pi_1(S^2) = 0$ (the 2-sphere is simply connected), not a non-contractible loop in $SU(2)$.

The $4\pi^2$ of the STF threshold involves null-direction winding, not spin-frame winding. The two phenomena are distinct and must not be conflated.

Appendix B: Explicit Verification of the Cross-Ratio Classification

Claim: For $\alpha = i\beta$ ($\beta > 0$, below threshold), the four points are in monotone real order.

Computation: $z_2 = e^{i(i\beta)} = e^{-\beta} \in (0,1)$, $z_4 = -e^{-\beta} \in (-1,0)$.

Order on $\mathbb{R}$: $z_3 = -1 < z_4 = -e^{-\beta} < z_2 = e^{-\beta} < z_1 = 1$. Strictly increasing: monotone. ✓

Cross-ratio: $\mathrm{cr}(i\beta) = \sec^2(i\beta/2) = 1/\cos^2(i\beta/2) = 1/\cosh^2(\beta/2) = \mathrm{sech}^2(\beta/2) \in (0,1)$. ✓

Claim: For real $\alpha \in (0,2\pi)$ (above threshold), the four points lie on the complex unit circle.

Computation: $|z_1| = |z_2| = |z_3| = |z_4| = 1$ trivially (since $|1| = |-1| = |e^{i\alpha}| = |-e^{i\alpha}| = 1$). They are distinct for $\alpha \neq 0, \pi$. ✓

Claim: The cross-ratio is 1 iff two pairs coincide.

$\mathrm{cr} = 1 \Leftrightarrow \sec^2(\alpha/2) = 1 \Leftrightarrow \cos(\alpha/2) = \pm 1 \Leftrightarrow \alpha \equiv 0 \pmod{2\pi}$. At $\alpha = 0$: $z_1 = z_2 = 1$, $z_3 = z_4 = -1$. ✓

Appendix C: The $\mathbb{CP}^3$ Transition Probability — Full Computation

This appendix records the complete computation behind the structural claim of Section 5.6 that the $\mathbb{CP}^3$ Fubini-Study transition probability cannot recover $P_{\mathrm{FS}} = (\tau/\tau^*)^{7/4}$.

Twistors. With $\lambda_\alpha = (1,1)^T$ and incidence relation $\mu^{\dot{\alpha}}(\tau) = ia(\tau)(1,1)^T$:

$Z_* = (1, 1, ia_*, ia_*)^T$, $\qquad Z(\tau) = (1, 1, ia(\tau), ia(\tau))^T$

$|Z_*|^2 = 2 + 2a_*^2$, $\qquad |Z(\tau)|^2 = 2 + 2a(\tau)^2$

Inner product split:

$$ Z_{*} \cdot \bar{Z} (\tau) = \underbrace{1 \cdot 1 + 1 \cdot 1}_{\lambda \text{ overlap} = 2} + \underbrace{\left(i a_{*} \right) \left(- i a (\tau) \right) + \left(i a_{*} \right) \left(- i a (\tau) \right)}_{\mu \text{ cross-term} = 2 a_{*} a (\tau)} $$

With $u = (\tau/\tau^*)^{1/4}$ so $a(\tau) = a_* u$:

$$ P_{\mathrm{FS}} (u) = \frac{\left(1 + a_{*}^{2} u \right)^{2}}{\left(1 + a_{*}^{2} \right) \left(1 + a_{*}^{2} u^{2} \right)} $$

Deviation identity (algebraic, exact):

$$ 1 - P_{\mathrm{FS}} (u) = \frac{a_{*}^{2} (u - 1)^{2}}{\left(1 + a_{*}^{2} \right) \left(1 + a_{*}^{2} u^{2} \right)} $$

Structural obstruction. Near $u = 1$ (threshold, $u = 1 - \epsilon$):

$$ 1 - P_{\mathrm{FS}} = \frac{a_{*}^{2} \epsilon^{2}}{\left(1 + a_{*}^{2} \right) \left(1 + a_{*}^{2} (1 - \epsilon)^{2} \right)} \approx \frac{\epsilon^{2}}{a_{*}^{2}} + O \left(\epsilon^{3} \right) $$

The ϵ^1 coefficient is identically zero. P_{FS} has a maximum at threshold; its deviation opens quadratically. The workspace formula $1 - (\tau/\tau^*)^{7/4}$ opens linearly. No reparametrisation of u can reconcile a quadratic zero with a linear zero at the same point.

Numerical check ($u = 1/2$, $a_* = 730$):

$$ 1 - P_{\mathrm{FS}} (1 / 2) = \frac{730^{2} \cdot (1 / 4)}{\left(1 + 730^{2} \right) \left(1 + 730^{2} / 4 \right)} \approx 1.87 \times 10^{-6} $$

$1 - (1/2)^7 = 1 - 1/128 \approx 0.992$

Ratio: $\approx 5 \times 10^5$ — approximately five to six orders of magnitude. The functions are structurally distinct across the entire domain, not only near threshold.

CITATION

```
@article{paz2026nullcone,
  author = {Paz, Z.},
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