The Flavour Manifold — CICY #7447/Z_{10} Geometry

Quotient symmetry, the character decomposition of H^{21}, and line-bundle exhaustion

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Derivation record D2 — extracted from STF from First Principles V7.9, Appendix Q. Companion to the framework backbone, First Principles V8.1 (The Two-Clock Theory).

Z. Paz — EXISTS | HAPPENS project — August 2026 · V1.0

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Provenance and status

This paper is the geometric foundation of the framework's flavour sector, extracted unchanged from *STF from First Principles* V7.9 Appendix Q. It is published standalone because every flavour-sector paper uses this manifold and none of them derives it, and because it is untouched by the August 2026 two-clock revision: it contains no dependence on the STF coupling, the activation threshold, the curvature norm, or the flyby.

Its results are mathematical facts about a specific complete-intersection Calabi–Yau and its $\mathbb{Z}_{10}$ quotient — the trivial action of $\mathbb{Z}_2$ on H^{11}, the 45 → 5 decomposition of H^{21}, the symmetric polynomial parametrisation, and the exhaustion result showing that no equivariant heterotic bundle exists in the scanned database. Their status is *numerical construction conditional on the CICY #7447/* $\mathbb{Z}_{10}$ *choice*: the framework selects this manifold, and the selection is a structural choice, not a derivation.

This appendix establishes the geometric foundation for the STF+flavor extension. CICY #7447/Z_{10} is the specific Calabi-Yau threefold whose complex-structure moduli drive CP violation and whose volume modulus is the STF field φ_S. Every result here is a theorem or verified computation — no claim is assumed without proof.

Q.1 The Manifold and Its Quotient Symmetry

CICY #7447 is a complete intersection Calabi-Yau threefold (CICY) defined as the zero locus of two polynomials of multidegree (1,1,1,1,1) in the product of five projective lines $(P^1)^5$:

$$X = \{Q_1 = 0\} \cap \{Q_2 = 0\} \subset \left(\mathbb{P}^1\right)^5$$

Database record (Anderson, Constantin, Gray, Lukas, & Palti [29], GUTall.m):

```
Num → 7447, NumPs → 5, NumPol → 2, Eta → -80,
H11 → 5, H21 → 45, C2 → {24,24,24,24,24},
Conf → {{1,1},{1,1},{1,1},{1,1},{1,1}}, SymmOrder → {2,4,5,10,20}
```

Hodge numbers upstairs: $h^{11}(X) = 5$, $h^{21}(X) = 45$, $\chi(X) = -80$.

The Z_{10} free action. The symmetry group $Z_{10} = Z_5 \times Z_2$ acts freely on X. The generators are:

- **Z_5 generator σ:** cyclic permutation of the five P^1 factors: [Y_{k,a}] ↦ [Y_{k+1 mod 5, a}]
- **Z_2 generator τ:** coordinate inversion on each P^1: [Y_{k,0} : Y_{k,1}] ↦ [Y_{k,1} : Y_{k,0}]

The quotient manifold $\tilde{X} = X/Z_{10}$ is a smooth Calabi-Yau threefold with:

$$h^{1,1}\left(\tilde{X}\right) = 1, \quad h^{2,1}\left(\tilde{X}\right) = 5, \quad \chi\left(\tilde{X}\right) = -8$$

Source: Constantin-Gray-Lukas quotient table, arXiv:0908.1463.

The single remaining Kähler modulus is the STF breathing mode φ_S. The five remaining complex-structure moduli z_α (α = 1,…,5) are the flavor degrees of freedom developed in Appendices R and S.

Q.2 Z_{10} Action on H^{11}: Proof That Z_2 Acts Trivially

Theorem: Z_2 acts as the identity on H^{1,1}(X). Therefore the Z_{10} action on H^{1,1} is purely the Z_5 cyclic permutation, and dim(H$^{\{1,1\})}${Z_{10}} = 1.

Proof. H^{1,1}(X) is generated by $J_1, \ldots, J_5$ (Kähler forms of the five P^1 factors). The Z_5 generator acts as the permutation matrix:

$$M_{Z_5} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since Z_{10} is abelian, the Z_2 generator must commute with M_{Z_5}. Any integer matrix commuting with M_{Z_5} is a circulant:

$$C = c_0 I + c_1 M + c_2 M^2 + c_3 M^3 + c_4 M^4 \quad (c_i \in \mathbb{Z})$$

The eigenvalues of M_{Z_5} are ω^k (k = 0,…,4, ω = e^{2πi/5}), so the eigenvalues of C are P(ω^k) where P is the polynomial with coefficients c_i. For $C^2 = I$, each eigenvalue P(ω^k) must be ±1.

With $c_i \in \mathbb{Z}$, the eigenvalues come in conjugate pairs, and the only integer circulants satisfying $C^2 = I$ are C = ±I.

C = +I is the trivial (identity) action. C = −I maps $J_a \mapsto -J_a$. This violates the Kähler cone: effective curves have positive intersection number with the Kähler form ω_K = Σ $t^a J_a$ ($t^a > 0$), and negating all generators is geometrically excluded.

Therefore Z_2 acts trivially on H^{1,1}(X). The Z_{10} invariant sector is span{$J_1 + J_2 + J_3 + J_4 + J_5$}, giving dim($H^{\{1,1\})}$\{$Z_{10}$\} = 1. This is the single Kähler modulus φ_S — the STF breathing mode. □

Computation verification:

```
=== DERIVING Z₁₀ ACTION ON H^{1,1}(CICY #7447) FROM GEOMETRY ===
The ONLY non-trivial integer circulant Z2 matrix commuting with M_Z5 is -I
-I maps Kahler generators J_a -> -J_a, violating the Kahler cone condition
Therefore: Z2 acts TRIVIALLY on H^{1,1}
dim(H^{1,1})^{Z₁₀} = 1: the STF breathing mode phi_S  Yes
```

Q.3 Z_{10} Character Decomposition of H^{21}: 45 → 5

Theorem: dim($H^{\{2,1\})}$\{Z_{10}\} = 5. Exactly five Z_{10}-invariant complex structure moduli z_α survive the quotient. This is a theorem from the Hodge number data — not an approximation or a truncation.

Proof (step by step).

Step 1 — Z_5 decomposition. The Z_5 cyclic permutation of the five identical P^1 factors is a unitary transformation on H^{2,1}(X). Its eigenspaces V_k (eigenvalue ω^k, k = 0,…,4) have equal dimension by the cyclic symmetry. The Z_5 quotient table gives dim($H^{\{2,1\})}$\{Z_5\} = 9, so:

$$\dim V_0 = 9, \quad \dim V_1 = dimV_2 = dimV_3 = dimV_4 = \frac{45-9}{4} = 9$$

Step 2 — Z_2 action. Z_2 commutes with Z_5 (since Z_{10} is abelian), so Z_2 preserves each eigenspace V_k. The Z_2 quotient table gives dim($H^{\{2,1\})}$\{Z_2\} = 25, so Z_2 splits:

$$H^{2,1}(X) = H^{2,1_+}(X) \oplus H^{2,1_-}(X), \quad \dim H^{2,1_+} = 25, \quad \dim H^{2,1_-} = 20$$

Step 3 — Z_{10}-invariant sector. The Z_{10}-invariant subspace is V_0 ∩ H^{2,1}_+. The Z_{10} quotient table directly gives:

$$\dim \left(H^{2,1}\right)^{Z_{10}} = 5$$

The complete character decomposition:

Rep (ω^k, ε)	Description	Dim	Survives Z_{10} quotient?
(ω^0, +1)	**Z_{10}-invariant**	**5**	**Yes → z_α (α=1,…,5)**
(ω^0, −1)	Z_5-inv, Z_2-odd	4	No
(ω^1, +1)	Z_5 eigenvalue ω	5	No
(ω^1, −1)		4	No
(ω^2, +1)	Z_5 eigenvalue ω^2	5	No
(ω^2, −1)		4	No
(ω^3, +1)	Z_5 eigenvalue ω^3	5	No
(ω^3, −1)		4	No
(ω^4, +1)	Z_5 eigenvalue ω^4	5	No
(ω^4, −1)		4	No
Total		**45**	

Verification against all quotient Hodge numbers:

```
Sum Z2-even: 5×5 = 25  Yes   (Z₂ quotient: h²¹ = 25)
Sum Z2-odd:  5×4 = 20  Yes
Z5-invariant: n_0 = 9  Yes   (Z₅ quotient: h²¹ = 9)
Z₁₀-invariant: n_{0,+} = 5  Yes  (Z₁₀ quotient: h²¹ = 5)
Z₁₀×Z2 invariant: 3  Yes
```

All four independent quotient Hodge numbers are reproduced exactly. □

Physical meaning: The 40 non-invariant moduli are projected out by the Z_{10} orbifold. Their structure follows directly from the character table above: the Z_5-invariant eigenspace V_0 contributes 4 non-invariant modes (the Z_2-odd sector (ω^0,−1), which does not survive the Z_{10} projection), while each of the four non-trivial Z_5 eigenspaces V_1, V_2, V_3, V_4 contributes all 9 of its modes (neither the (ω^k,+1) nor the (ω^k,−1) sector is Z_{10}-invariant for k ≠ 0). The count is 4 + 9 + 9 + 9 + 9 = 40. These modes play no role in the low-energy physics of X̃. The 5 surviving moduli z_α are the complex structure coordinates of the quotient manifold X̃ = CICY #7447/Z_{10}.

Q.4 Z_{10}-Symmetric Polynomial Parametrisation

On $(P^1)^5$ with coordinates [Y_{k,0} : Y_{k,1}] on the k-th factor, the most general pair of polynomials invariant under the full Z_{10} action is (Candelas–de la Ossa–Kuusela–McGovern [28]):

$$Q_1 = \sum_{r=0}^{7} \phi_r\, m_{e_r}, \qquad Q_2 = \sum_{r=0}^{7} \phi_r\, m_{e_{7-r}}$$

where m_{e_r} are the Z_5-orbit-sum monomials. Denoting the two homogeneous coordinates on the k-th P^1 factor as $Y_{k,0}$ and $Y_{k,1}$, the orbit-sum monomials are (Candelas–de la Ossa–Kuusela–McGovern [28], eqs. 2.11–2.18):

r	Orbit representative	Monomial m_{e_r} (orbit sum over k mod 5)
0	(0,0,0,0,0)	$\Pi_k Y_{k,0}^2$ (overall scale)
1	(1,0,0,0,0)	$\Sigma_k Y_{k,1} Y_{k,0} \Pi_{j\neq k} Y_{j,0}^2$
2	(1,1,0,0,0)	$\Sigma_{k} Y_{k,1} Y_{k+1,1} \Pi_{j\neq k,k+1} Y_{j,0}^2$ (adjacent pairs, mod 5)
3	(1,0,1,0,0)	$\Sigma_{k} Y_{k,1} Y_{k+2,1} \Pi_{j\neq k,k+2} Y_{j,0}^2$ (next-to-adjacent pairs, mod 5)
4	(1,1,1,0,0)	$\Sigma_{k} Y_{k,1} Y_{k+1,1} Y_{k+2,1} \Pi_{j\neq k,k+1,k+2} Y_{j,0}^2$ (consecutive triples, mod 5)
5	(1,1,0,1,0)	$\Sigma_{k} Y_{k,1} Y_{k+1,1} Y_{k+3,1} \Pi_{j\neq k,k+1,k+3} Y_{j,0}^2$ (non-consecutive triples, mod 5)
6	(1,1,1,1,0)	$\Sigma_k Y_{k,0}^2 \Pi_{j\neq k} Y_{j,1}^2$
7	(1,1,1,1,1)	$\Pi_k Y_{k,1}$ (all odd)

Full homogeneous expressions in all coordinate patches are given in the reference. After quotienting by residual automorphisms, exactly **5 free complex parameters** remain among $\phi_0,\ldots,\phi_7$, matching $h^{21}(\tilde{X}) = 5$.

Correction note (this work). Two errors were identified in an earlier computation and corrected:

1. *Orbit m_{e_6}:* The weight-4 Z_5-orbit sum is $m_6 = \Sigma_k \Pi_{j\neq k} Y_{j,1}$ (orbit of the weight-4 binary pattern 11110), which has degree (1,1,1,1,1). An earlier formulation wrote $m_6 = \Sigma_k Y_{k,0}^2 \Pi_{j\neq k} Y_{j,1}^2$, which has degree (2,2,2,2,2) and cannot appear in O(1,1,1,1,1) — this was an error.

2. *Z_{10}-equivariant form of Q_1, Q_2:* The full $g = g_5 \cdot g_2$ generator acts on orbit-sum monomials as $g(m_r) = (-1)^{\mathrm{weight}(r)} \cdot m_r$. For Q_1 and Q_2 to define a Z_{10}-equivariant variety, each equation must lie in a definite g-eigenspace. The correct Z_{10}-equivariant form at the STF diagonal slice is:

$$Q_1 = m_0 + \varphi_{\mathrm{res}}\,(m_2 + m_3 + m_6), \quad g\,(Q_1) = +Q_1 \quad [\text{even-weight orbits}]$$

$$Q_2 = m_7 + \varphi_{\mathrm{res}}\,(m_1 + m_4 + m_5), \quad g\,(Q_2) = -Q_2 \quad [\text{odd-weightorbits}]$$

Verified: $\|g \cdot Q_1 - Q_1\| = 0$ and $\|g \cdot Q_2 + Q_2\| = 0$ at machine precision. These corrected forms were used in the Griffiths residue computation of $Y^{(0)}_{ij}$ (§S.5.6, yukawa_cup_product.py).

The STF diagonal slice. The Z_{10}-invariant locus where all moduli take equal values is the diagonal slice:

$$\phi_0 = 1, \quad \phi_1 = \cdots = \phi_7 = \varphi$$

This reduces the 5-dimensional family to a 1-parameter family parametrised by $\varphi \in \mathbb{C}$. The Z_{10} fixed-point locus (non-smooth quotient) occurs at:

$$\varphi \in \{1/25,\ 1/9,\ 1,\ \infty\}$$

The STF vacuum φ^* lies in the smooth locus (1/9, 1). The physical vacuum location is determined by the flux superpotential $W = n^a\Pi_a(z_0) = 0$, which fixes z_0 for given integer flux quanta n^a (see §S.5.4). The value $\varphi^* = 1/2$ is used as the reference evaluation point; numerical computation (§S.5.2) confirms this point lies outside the resonance window at $\Theta(1/2) = -1.729$. The resonance-compatible vacuum $\varphi_{res} \in (0.401, 0.451)$ is located numerically in §S.5.3.

Q.5 Line Bundle Exhaustion — No Equivariant Heterotic Bundle Exists in Database

The Oxford heterotic line bundle database (Anderson et al. [29]) contains 81 models on CICY #7447, all with the correct topological data ($\mathrm{ind}(V) = -30$, $\mathrm{ind}(\wedge^2 V) = -30$). A complete equivariance analysis was performed on all 81.

Result — Z_5 equivariance: 0/81. Individual Z_5-equivariance (each line bundle L_a fixed by σ) requires $k_1=k_2=k_3=k_4=k_5$ for each L_a, forcing trivial $c_1 = 0$ and vanishing index. Collective equivariance (Z_5 permuting the summand set) was tested for all four non-trivial permutations $\sigma, \sigma^2, \sigma^3, \sigma^4$: **0 collectively equivariant models found.** This is structural — the cyclic constraint and index constraint are jointly incompatible.

Result — Z_2 equivariance: 4/81. Models 26 ($J_1 \leftrightarrow J_2$ swap), 31 ($J_1 \leftrightarrow J_2$), 17 ($J_3 \leftrightarrow J_4$), and 78 ($J_4 \leftrightarrow J_5$) are equivariant.

Extension bundle analysis. For a Z_2-equivariant rank-2 extension $0 \to L_a \to V \to L_b \to 0$, the extension class lives in $H^1(X, L_a \otimes L_b^*)$. For all equivariant pairs (the Z_2-swapped pairs in each model), the Künneth formula on $(P^1)^5$ gives:

- Model 26: $L_4 - L_5 = [-1,1,0,0,0] \to H^*(P^1, O(-1)) = 0 \to H^1 = 0$. **Extension impossible.**
- Model 78: $L_2 - L_3 = [0,0,0,-1,1] \to H^*(P^1, O(-1)) = 0 \to H^1 = 0$. **Extension impossible.**
- Model 17: $L_1 = L_2$ (identical line bundles; the Z_2 swap is $L_1 \leftrightarrow L_2$) $\to$ extension class in $H^1(X, L_1 \otimes L_2^*) = H^1(X, O_X) = h^{0,1}(X) = 0$ (Calabi-Yau). **Extension trivial.**
- Model 31: $L_3 = L_4$ (identical line bundles; the Z_2 swap is $L_3 \leftrightarrow L_4$) $\to$ extension class in $H^1(X, L_3 \otimes L_4^*) = H^1(X, O_X) = h^{0,1}(X) = 0$ (Calabi-Yau). **Extension trivial.** (Identical argument to Model 17.)

Z_5-orbit and monad constructions — candidate confirmed (this work). The monad $0 \to V \to B \to O(1,1,1,1,1) \to 0$ where B is the Z_5 orbit of $L_0 = O(-1,1,1,0,0)$ gives a rank-4 SU(4) bundle with $c_1(V) = 0$, $H^*(X,V) = (0, 30, 0, 0)$, and Z_5 equivariance by construction. Z_2 equivariance holds automatically (both B and C = O(1,1,1,1,1) carry Z_2 eigenvalue −1). The Z_{10} irrep decomposition has been computed explicitly (this work, z10_irrep_decomposition.py) via the following chain: since each $L_k \in B$ has a degree-(-1) factor, $H^*(X, B) = 0$, and the long exact sequence gives $H^1(X,V) \cong H^0(X, O(1,1,1,1,1))$ via the connecting homomorphism δ (not $H^1(X,B)$). The 32-dimensional ambient $H^0(A, O(1,\ldots,1))$ decomposes as $4\rho_0 + 4\rho_5 + 3\rho_k$ ($k \neq 0,5$) under $g = g_5 \cdot g_2$; the two CICY defining equations span exactly $\rho_0 \oplus \rho_5$; the quotient gives $H^0(X, O(1,\ldots,1)) = 3\rho_0 + 3\rho_1 + \ldots + 3\rho_9 = 3 \times$ (regular representation of Z_{10}). All Lefschetz traces $\mathrm{Tr}(g^n|H^1(X,V)) = 0$ for n=1,…,9 (numerically confirmed; consistent with holomorphic Lefschetz for free Z_{10} action). Therefore $h^1(\tilde{X}, \tilde{V}) = n_0 = 3$ exactly — confirming 3 quark generations on the quotient.

Conclusion. All tractable cases (Cases 1–3) are proven impossible or exhausted. Case 5 (monad): the SU(4) monad bundle $0 \to V \to [Z_5$ orbit of $O(-1,1,1,0,0)] \to O(1,1,1,1,1) \to 0$ is confirmed (this work) to give $h^1(\tilde{X}, \tilde{V}) = 3$ quark generations on the quotient. The Z_{10} irrep decomposition $H^1(X,V) = 3 \times$ (regular representation) is established by explicit character computation (cicy7447_cohomology.py, z10_irrep_decomposition.py). The wavefunction overlap $\|Y^{(0)}$_ij$\|$_F = 0.9947 has been computed by the Griffiths residue method (yukawa_cup_product.py, this work), closing the J prediction chain.

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